Stability Enhancement by Boundary Control in 2D Channel Flow*

Andras Balogh , Wei–Jiu Liu and Miroslav Krstic

Department of MAE

University of California at San Diego

La Jolla, CA 92093-0411

fax: 858-822-3107

858–822–2406 858–822–2406 858-822-1374

abalogh@ucsd.edu weiliu@mae.ucsd.edu krstic@ucsd.edu

http://mae.ucsd.edu/research/krstic

Abstract

In this paper we stabilize the parabolic equilibrium profile in a 2D channel flow using actuators and sensors only at the wall. The control of channel flow was previously considered by Speyer and coworkers, and Bewley and coworkers, who derived feedback laws based on linear optimal control, and implemented by wall–normal actuation. With an objective to achieve global Lyapunov stabilization, we arrive at a feedback law using tangential actuation (using teamed pairs of synthetic jets or rotating disks) and only local measurements of wall shear stress, allowing to embed the feedback in MEMS hardware, without need for wiring. This feedback is shown to guarantee global stability in at least H^2 norm, which by Sobolev's embedding theorem implies continuity in space and time of both the flow field and the control (as well as their convergence to the desired steady state). The theoretical results are limited to low values of Reynolds number, however, we present simulations that demonstrate the effectiveness of the proposed feedback for values five order of magnitude higher.

Key words. 2D channel flow, Navier–Stokes equations, tangential velocity actuation, boundary feedback, Lyapunov stability.

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1 Introduction

In this article we address the problem of boundary control of a viscous incompressible fluid flow in a 2D channel. Great advances have been made on this topic by Speyer and coworkers [14, 38, 39], Bewley and coworkers [4, 5, 7], and others employing optimal control techniques in the CFD setting. Equally impressive progress was made on the topic of controllability of Navier–Stokes equations , which is, in a sense, a prerequisite to all other problems.

Our objective in this paper is to globally *stabilize* the parabolic equilibrium profile in channel flow. This objective is different than the efforts on optimal control [2, 3, 16, 18, 19, 20, 21, 26, 30, 31, 33, 34, 36, 60] or controllability [10, 11, 13, 17, 22, 23, 24, 25, 27, 28, 29, 35] of Navier–Stokes equations. Optimal control of nonlinear equations such as Navier–Stokes is not solvable in closed form, forcing the designer to either linearize or use computationally expensive finite–horizon model–predictive methods. Controllability–based solutions, while a prerequisite to all other problems, are not robust to changes in the initial data and model inaccuracies. The stabilization objective indirectly addresses the problems of turbulence and drag reduction, which are explicit in optimal control or controllability studies. Coron's [12] result on stabilization of Euler's equations is the first result that directly addresses flow stabilization. Concerning other nonlinear PDEs with convective nonlinearities, examples of stabilization and controllability studies can be found in [45, 54, 55] for the 1D Korteweg–de Vries equation.

The boundary feedback control we derive in this paper is fundamentally different from those in [14, 38, 39, 4, 5, 7], which use *wall normal* blowing and suction. Our analysis motivated by Lyapunov stabilization results in *tangential* velocity actuation. Tangential actuation is technologically feasible. The work on synthetic jets of Glezer [59] shows that a teamed up pair of synthetic jets can achieve an angle of 85° from the normal direction with the same momentum as wall normal actuation. The patent of Keefe [43] provides the means for generating tangential velocity actuation using arrays of rotating disks.

An implementational advantage in our result is that, while it uses only the measurement of wall shear stress as in the previous efforts, it employs it in a *decentralized* fashion. This means that the feedback law can be embedded into the MEMS hardware (without need for wiring).

The most notable contribution of this paper is in the form of stability it achieves. Previous studies of the stability problem for uncontrolled Navier–Stokes equations were in the case of homogeneous Dirichlet boundary conditions [53, 61], periodic boundary conditions [62] or the domain being the whole space [32, 40, 41, 42, 46, 52, 58, 63, 64, 65]. In the case of bounded domains, these stability results were estimated in terms of L^2 or L^p norm and it is rare to see H^1 stability, especially H^2 stability. We obtain global H^2 stability (i.e., for arbitrarily large H^2 initial data) which, in turn, ensures the continuity of the flow field.

The only limitation in our result is that it is guaranteed only for sufficiently low values of the Reynolds number. In simulations we demonstrate that the control law has a stabilizing effect far beyond the value required in the theorem (five or more orders of magnitude).

Our feedback is not limited to 2D channel flows. It applies equally well to 3D for L^2 stabilization. However, higher forms of global stability are impossible to prove due to the same technical obstacles that prevent proving uniqueness of solutions in 3D Navier–Stokes equations. Numerical evaluation of this feedback in 3D channel flow is nontrivial and is a topic of future research.

The paper is organized as follows. We formulate our problem in Section 2 and design boundary feedback laws in Section 3. In order to state our main results, we first present some mathematical preliminaries in Section 4 and then state the results in Section 5. In order to prove the results, we need technical lemmas which are presented in Section 6. With these technical lemmas at hand, we prove our results in Section 7 by employing Lyapunov techniques and Galerkin's methods. Finally, in Section 8, we give numerical demonstrations that strengthen our theoretical results.

2 Problem Statement

The channel flow can be described by the 2D Navier–Stokes equations

$$\begin{cases} \mathbf{W}_t - \nu\Delta\mathbf{W} + (\mathbf{W}\cdot\nabla)\,\mathbf{W} + \nabla P = 0\,, & 0 < x < 1,\, 0 < y < l,\, t > 0, \\ \mathrm{div}\mathbf{W} = 0\,, & 0 < x < 1,\, 0 < y < l,\, t > 0, \end{cases} \tag{2.1}$$

where $\mathbf{W} = \mathbf{W}(x,y,t) = (U(x,y,t), V(x,y,t))^T$ represents the velocity vector of a particle at (x,y) and at time t, $P = P(x,y,t)$ is the pressure at (x,y) and at time t, $\nu > 0$ is the kinematic viscosity and the positive constant l represents the width of the channel. Our goal is to regulate the flow to the parabolic equilibrium profile (see Figure 2.1)

$$\overline{U}\,(y) = \frac{a}{2\nu}y\,(l-y)\ , \tag{2.2}$$

$$\overline{V} = 0\,, \tag{2.3}$$

$$\overline{P}\,(x) = -ax + b\,, \tag{2.4}$$

where $a = \overline{P}\,(0) - \overline{P}\,(1) \geq 0$ and $b = \overline{P}\,(0) \geq 0$ are constants. This profile is obtained as a fixed point of system (2.1).

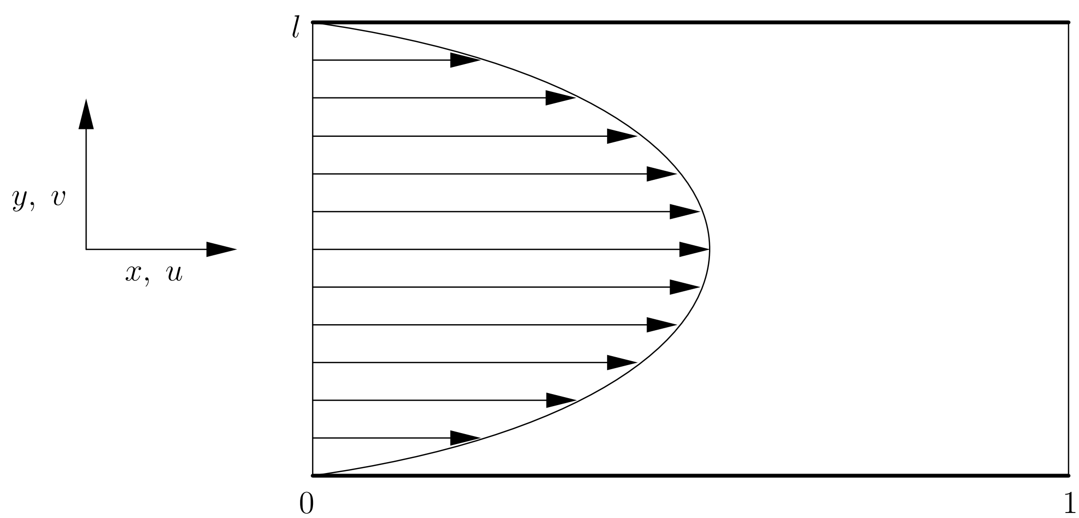

Figure 2.1: 2D channel flow

To motivative our problem, let us consider the vorticity

$$\omega\,(x,y,t) = U_y\,(x,y,t) - V_x\,(x,y,t)\ . \tag{2.5}$$

With (2.2) and (2.3), we get the equilibrium vorticity as

$$\overline{\omega}\,(y) = \overline{U}'\,(y) - \overline{V}' = \frac{a}{2\nu}\,(l-2y)\ . \tag{2.6}$$

Suppose the vorticity at the walls is kept at its equilibrium values

$$\omega(x,0,t) = \overline{\omega}(0) , \quad \omega(x,l,t) = \overline{\omega}(l) , \tag{2.7}$$

and the wall–normal component of the velocity at the walls is zero:

$$V(x,0,t) = 0 , \quad V(x,l,t) = 0 . \tag{2.8}$$

The objective of these no–feedback boundary conditions might be the reduction of near–wall vorticity fluctuations. These boundary conditions imply

$$U_y(x,0,t) = \omega(x,0,t) + V_x(x,0,t) = \frac{al}{2\nu} , \tag{2.9}$$

$$U_y(x,l,t) = \omega(x,l,t) + V_x(x,l,t) = -\frac{al}{2\nu} . \tag{2.10}$$

Under the boundary conditions (2.8)–(2.10), the Stokes equations

$$-\nu\Delta\mathbf{W} + (\mathbf{W}\cdot\nabla)\mathbf{W} + \nabla P = 0 , \tag{2.11}$$

$$\mathrm{div}\mathbf{W} = 0 \tag{2.12}$$

has a solution

$$U = \overline{U}(y) + c , \tag{2.13}$$

$$V = \overline{V} , \tag{2.14}$$

$$P = \overline{P}(x) , \tag{2.15}$$

with an arbitrary constant C. This shows that under the boundary control (2.8)–(2.10) our objective of regulation to the equilibrium solution (2.2)–(2.3) can not be achieved. In more precise words, this solution is not asymptotically stable, and it can at best be marginally stable (with an eigenvalue at zero). To achieve *asymptotic* stabilization, in the next section we propose a feedback law which modifies the boundary condition (2.7).

3 Boundary Feedback Laws

In order to prepare for regulating the flow to the parabolic equilibrium profile (2.2)–(2.3), we set

$$u = U - \overline{U} , \tag{3.1}$$

$$v = V , \tag{3.2}$$

$$p = P - \overline{P} . \tag{3.3}$$

Then equation (2.1) becomes

$$\begin{cases} u_t - \nu\Delta u + uu_x + vu_y + \overline{U}u_x + \overline{U}'v + p_x = 0 , & 0 < x < 1,\, 0 < y < l,\, t > 0 , \\ v_t - \nu\Delta v + uv_x + vv_y + \overline{U}v_x + p_y = 0 , & 0 < x < 1,\, 0 < y < l,\, t > 0 , \\ u_x + v_y = 0 , & 0 < x < 1,\, 0 < y < l,\, t > 0 , \\ u(x,y,0) = u_0,\, v(x,y,0) = v_0 , & 0 < x < 1,\, 0 < y < l, \end{cases} \tag{3.4}$$

To avoid dealing with an infinitely long channel, we assume that u, v, v_x and p are *periodic in the x–direction*, i.e.,

$$u(0,y,t) = u(1,y,t), \; v(0,y,t) = v(1,y,t), \qquad 0 < y < l, \; t > 0, \tag{3.5}$$

$$v_x(0,y,t) = v_x(1,y,t), \; p(0,y,t) = p(1,y,t), \qquad 0 < y < l, \; t > 0. \tag{3.6}$$

Our boundary control is applied via boundary conditions

$$\begin{cases} u(x,0,t) = ku_y(x,0,t), & 0 < x < 1, \; t > 0, \\ u(x,l,t) = -ku_y(x,l,t), & 0 < x < 1, \; t > 0, \\ v(x,0,t) = 0, & 0 < x < 1, \; t > 0, \\ v(x,l,t) = 0, & 0 < x < 1, \; t > 0, \end{cases} \tag{3.7}$$

where k is a positive constant. The physical implementation of this boundary condition is

$$U(x,0,t) = k\left[U_y(x,0,t) - \frac{al}{2\nu}\right], \tag{3.8}$$

$$U(x,l,t) = -k\left[U_y(x,l,t) + \frac{al}{2\nu}\right], \tag{3.9}$$

$$V(x,0,t) = 0, \tag{3.10}$$

$$V(x,l,t) = 0. \tag{3.11}$$

This means that we are actuating the flow velocity at the wall *tangentially*. Only the sensing of the wall shear stress $U_y(x,0,t)$ and $U_y(x,l,t)$ (at the respective points of actuation) is needed. The action of this feedback is pictorially represented in Figure 3.1. The condition (3.8) and (3.9) can be also written as

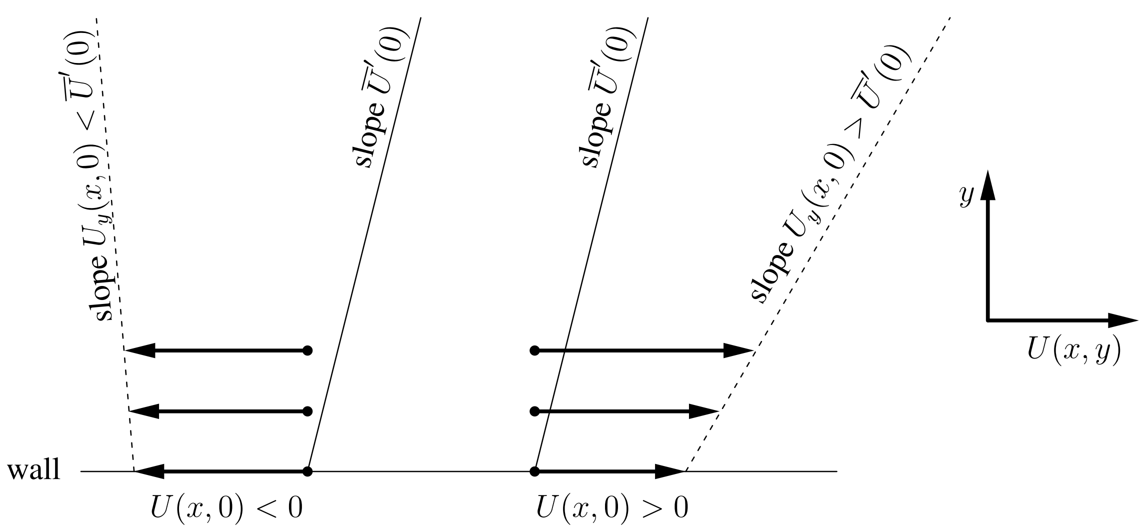

Figure 3.1: Tangential velocity actuation

$$U(x,0,t) = k\left[\omega(x,0,t) - \overline{\omega}(0)\right], \tag{3.12}$$

$$U(x,l,t) = -k\left[\omega(x,l,t) - \overline{\omega}(l)\right]. \tag{3.13}$$

In the next sections we shall see that this control law achieves global asymptotic stabilization, whereas, as we saw in Section 2, the control law (2.7) is not asymptotically stabilizing.

4 Mathematical Preliminaries

Let $\Omega = (0,1) \times (0,l)$. In what follows, $H^s(\Omega)$ denotes the usual Sobolev space (see [1, 49]) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. We denote by $\widetilde{H^s}(\Omega)$ the space of the restrictions to Ω of functions which are in $H^s_{loc}(\mathbb{R}^2)$, i.e., $u|_O \in H^s(\Omega)$ for every open bounded set O, and which are periodic in the x–direction:

$$u(x,y) = u(x+1,y). \tag{4.1}$$

The tilde sign will refer to this periodicity in the case of other classical function spaces too.

We shall often be concerned with 2–dimensional vector function spaces and use the following notation to denote them:

$$\begin{aligned}
\widetilde{\mathbf{L}}^2 &= \left\{\widetilde{L}^2(\Omega)\right\}^2, && (4.2)\\
\widetilde{\mathbf{H}}^1 &= \left\{\widetilde{H}^1(\Omega)\right\}^2, && (4.3)\\
\widetilde{\mathbf{H}}^2 &= \left\{\widetilde{H}^2(\Omega)\right\}^2, && (4.4)\\
\widetilde{\mathcal{V}} &= \left\{\varphi \in \widetilde{C}^\infty(\Omega) \,:\, \varphi(x,\cdot) \in C_0^\infty((0,l)) \quad \forall x \in [0,1]\right\}, && (4.5)\\
\widetilde{\mathbf{V}} &= \left\{(u,v) \in \widetilde{\mathbf{H}}^1 \,:\, u_x + v_y = 0 \text{ in } \Omega,\ v(x,0) = v(x,l) = 0\right\}, && (4.6)\\
\widetilde{\mathbf{H}} &= \text{the closure of } \widetilde{\mathbf{V}} \text{ in } \widetilde{\mathbf{L}}^2. && (4.7)
\end{aligned}$$

The various norms of these spaces are respectively defined by

$$\begin{aligned}
\|\mathbf{w}\|_{\widetilde{\mathbf{L}}^2} &= (\mathbf{w},\mathbf{w})^{1/2}, && (4.8)\\
\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^1} &= \left(\|\mathbf{w}\|^2_{\widetilde{\mathbf{L}}^2} + \|\nabla u\|^2_{\widetilde{\mathbf{L}}^2} + \|\nabla v\|^2_{\widetilde{\mathbf{L}}^2}\right)^{1/2}, && (4.9)\\
\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^2} &= \left(\|\mathbf{w}\|^2_{\widetilde{\mathbf{H}}^1} + \|\nabla u_x\|^2_{\widetilde{\mathbf{L}}^2} + \|\nabla u_y\|^2_{\widetilde{\mathbf{L}}^2} + \|v_x\|^2_{\widetilde{\mathbf{L}}^2} + \|\nabla v_y\|^2_{\widetilde{\mathbf{L}}^2}\right)^{1/2}, && (4.10)\\
\|\mathbf{w}\|_{\widetilde{\mathbf{V}}} &= ((\mathbf{w},\mathbf{w}))^{1/2}, && (4.11)
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product of $\widetilde{\mathbf{L}}^2$ and $((\cdot,\cdot))$ denotes the inner product of $\widetilde{\mathbf{V}}$ defined by

$$((\mathbf{w},\mathbf{\Phi})) = \int_0^l \int_0^1 Tr\left\{\nabla\mathbf{w}^T \nabla\mathbf{\Phi}\right\} dxdy + \frac{1}{k}\int_0^1 \left(u(x,0)\,\xi(x,0) + u(x,l)\,\xi(x,l)\right) dx\,, \tag{4.12}$$

for all $\mathbf{w} = (u,v)$, $\mathbf{\Phi} = (\xi,\eta) \in \widetilde{\mathbf{V}}$.

Let X be a Banach space. We denote by $C^l([0,T];X)$ the space of l times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$.

Definition 4.1. *A function* $\mathbf{w} = (u,v) \in L^2([0,T];\widetilde{\mathbf{V}})$ *is a* weak solution *of system (3.4)–(3.7) if*

$$\frac{d}{dt}(\mathbf{w},\mathbf{\Phi}) + \nu\,((\mathbf{w},\mathbf{\Phi})) + ((\mathbf{w}\cdot\nabla)\,\mathbf{w},\mathbf{\Phi}) + \left(\overline{U}\mathbf{w}_x,\mathbf{\Phi}\right) + \left(\overline{U}'v,\xi\right) = 0 \tag{4.13}$$

is satisfied for all $\mathbf{\Phi} = (\xi,\eta) \in \widetilde{\mathbf{V}}$ *and* $\mathbf{w}(x,y,0) = \mathbf{w}_0(x,y)$ *for all* $(x,y) \in \Omega$.

5 The Results

Theorem 5.1. *Suppose that*[1]

$$\nu > \sqrt{\frac{al^3}{4}} \quad \text{and} \quad 0 < k < l/2, \tag{5.1}$$

and denote

$$\sigma = \frac{\nu}{l^2} - \frac{al}{4\nu} > 0. \tag{5.2}$$

Then there exists a positive constant $c > 0$ independent of $\mathbf{w}_0$ such that the following statements are true for all $t \geq 0$ for the system (3.4) with periodic conditions (3.5)–(3.6) and boundary control (3.7).

1. *For arbitrary initial data $\mathbf{w}_0(x) \in \widetilde{\mathbf{H}}$, there exists a unique weak solution $\mathbf{w} \in L^2\left([0,\infty); \widetilde{\mathbf{V}}\right) \cap C\left([0,\infty); \widetilde{\mathbf{L}}^2\right)$ that satisfies the following global–exponential stability estimate:*

$$\|\mathbf{w}(t)\| \leq \|\mathbf{w}_0\| \, e^{-\sigma t}. \tag{5.3}$$

2. *For arbitrary initial data $\mathbf{w}_0(x) \in \widetilde{\mathbf{V}}$, there exists a unique weak solution $\mathbf{w} \in L^2\left([0,\infty); \widetilde{\mathbf{H}}^2 \cap \widetilde{\mathbf{V}}\right) \cap L^\infty\left([0,\infty); \widetilde{\mathbf{V}}\right)$ that satisfies the following global–asymptotic and semiglobal–exponential stability estimate:*

$$\|\mathbf{w}\,(t)\,\|_{\widetilde{\mathbf{H}}^1} \leq c\|\mathbf{w}_0\|_{\widetilde{\mathbf{H}}^1} \exp(c\|\mathbf{w}_0\|^4_{\widetilde{\mathbf{H}}^1}) e^{-\sigma t/2}. \tag{5.4}$$

3. *For arbitrary initial data $\mathbf{w}_0(x) \in \widetilde{\mathbf{H}}^2 \cap \widetilde{\mathbf{V}}$ compatible with the control (3.7), there exists a unique weak solution $\mathbf{w} \in C^1\left([0,\infty); \widetilde{\mathbf{L}}^2\right) \cap C\left([0,\infty); \widetilde{\mathbf{H}}^2 \cap \widetilde{\mathbf{V}}\right)$ that satisfies the following global–asymptotic and semiglobal–exponential stability estimate:*

$$\|\mathbf{w}\,(t)\,\|_{\widetilde{\mathbf{H}}^2} \leq c\|\mathbf{w}_0\|_{\widetilde{\mathbf{H}}^2} \exp(c\|\mathbf{w}_0\|^4_{\widetilde{\mathbf{H}}^2}) e^{-\sigma t/2}. \tag{5.5}$$

   The bound of the form (5.5) also applies to $\|\mathbf{w}_t\,(t)\|$, $\|\nabla p(t)\|$ and $\max_{(x,y)\in\overline{\Omega}} |\mathbf{w}(x,y,t)|$.

In all of the above cases solutions depend continuously on the initial data in the L^2–norm and the existence, uniqueness and regularity statements hold for any $\nu > 0$ and $k > 0$ over finite time intervals.

Remark 5.1. Weak solutions satisfying the regularity stated in parts 2 and 3 of Theorem 5.1 are called strong solutions in the literature. Part 3 of Theorem 5.1 means, in particular, that

1. The control inputs $u(x,0,t)$ and $u(x,l,t)$ are bounded and go to zero as $t \to \infty$.
2. The regularity statement implies that $\mathbf{w}(x,y,t)$ is continuous in all three arguments. This observation has an important practical consequence: the tangential velocity actuation at nearby points on the wall will be in the same direction.

Remark 5.2. If the viscosity $\nu \leq \sqrt{\frac{al^3}{2}}$, the problem of boundary control remains open. The methods presented in this paper can not be applied to this case and a radically different method needs to be developed.

[1] Note that this condition is equivalent to the requirement that the Reynolds number be smaller than $1/8$.

6 Technical Lemmas

In this section, we establish technical lemmas which are the key to proving our main results.

Since $\widetilde{\mathbf{H}}$ is a closed subspace of $\widetilde{\mathbf{L}}^2$, we have the orthogonal decomposition

$$\widetilde{\mathbf{L}}^2 = \widetilde{\mathbf{H}} \oplus \widetilde{\mathbf{H}}^{\perp}, \tag{6.1}$$

where $\widetilde{\mathbf{H}}^{\perp}$ denotes the orthogonal complement of $\widetilde{\mathbf{H}}$. Let $\mathcal{P}$ denote the projection from $\widetilde{\mathbf{L}}^2$ onto $\widetilde{\mathbf{H}}$. We define the linear operator A on $\widetilde{\mathbf{H}}$ as

$$A\mathbf{w} = -\mathcal{P}\Delta\mathbf{w}\,, \tag{6.2}$$

with the domain $D(A)$

$$D(A) = \left\{ \mathbf{w} = (u, v) \in \widetilde{\mathbf{H}}^2 \cap \widetilde{\mathbf{V}} \,:\, u(x, 0) = k u_y(x, 0)\,,\ u(x, l) = -k u_y(x, l) \right\}. \tag{6.3}$$

We first give some basic properties of the subspaces $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{H}}^{\perp}$ and the operator A. These properties are similar to the classical results in the cases with homogeneous Dirichlet boundary condition (see, e.g., [61, Chap.I, Sect.1], [9, Chap.4]) and periodic boundary condition (see, e.g., [62, Chap.2]). Thus, their proofs are also similar, however, for completeness, we give brief proofs. The following lemma shows that (6.1) is in fact the so called Helmholtz decomposition of $\widetilde{\mathbf{L}}^2$.

Lemma 6.1. *The subspaces $\widetilde{\mathbf{H}}$ and $\widetilde{\mathbf{H}}^{\perp}$ can be characterized as follows:*

$$\widetilde{\mathbf{H}}^{\perp} = \left\{ \mathbf{w} \in \widetilde{\mathbf{L}}^2 \,:\, \mathbf{w} = \nabla p,\ p \in \widetilde{H}^1(\Omega) \right\}, \tag{6.4}$$

$$\widetilde{\mathbf{H}} = \left\{ \mathbf{w} = (u, v) \in \widetilde{\mathbf{L}}^2 \,:\, \mathrm{div}\mathbf{w} = 0,\ v(x, 0) = v(x, l) = 0 \right\}. \tag{6.5}$$

Proof. The proof of (6.5) is the same as the proof of Theorem 1.4 in [61, p.15]. We include the proof of (6.4) which is based on the proof of Theorem 1 in [47, p.27].

Let $\mathbf{w} = (u, v)$ belong to the space on the right hand side of (6.4). Then for all $\mathbf{z} = (\psi, \xi) \in \widetilde{\mathbf{V}}$ we have, using integration by parts,

$$\int_0^l \int_0^1 (u\psi + v\xi)\, dxdy = \int_0^l \int_0^1 (p_x\psi + p_y\xi)\, dxdy = 0\,. \tag{6.6}$$

Since $\widetilde{\mathbf{V}}$ is dense in $\widetilde{\mathbf{H}}$, we deduce that $\mathbf{w} \in \widetilde{\mathbf{H}}^{\perp}$.

Conversely, if $\mathbf{w} = (u, v) \in \widetilde{\mathbf{H}}^{\perp}$, then

$$\int_0^l \int_0^1 (u\psi + v\xi)\, dxdy = 0, \quad \forall \mathbf{z} = (\psi, \xi) \in \widetilde{\mathbf{V}}\,. \tag{6.7}$$

Let $\omega_\rho(x, y)$ denote a mollifier. For $\varphi \in \widetilde{\mathcal{V}}$, we denote by φ_ρ its average:

$$\varphi_\rho(x, y) = \int_{\mathbb{R}^2} \omega_\rho(x - s, y - \tau)\, \varphi(x, \tau)\, dsd\tau\,. \tag{6.8}$$

If ρ is small enough, then φ_ρ is well–defined on $\Omega_\rho = [0,1] \times [\rho, l-\rho]$, it is periodic in the x–direction and vanishes with its derivatives on the horizontal lines $y=\rho$ and $y = l-\rho$. Hence

$$\mathbf{z} = (\varphi_{\rho y}, -\varphi_{\rho x}) \in \widetilde{\mathbf{V}}\,. \tag{6.9}$$

Thus, we have

$$0 = \int_0^l \int_0^1 (u\varphi_{\rho y} - v\varphi_{\rho x})\; dxdy = \int_0^l \int_0^1 (u_\rho \varphi_y - v_\rho \varphi_x)\; dxdy = \int_0^l \int_0^1 (v_{\rho x} - u_{\rho y})\; dxdy\,, \tag{6.10}$$

where the functions u_ρ and v_ρ are defined on Ω_ρ and are the averages of u and v respectively. Since $\varphi \in \widetilde{\mathcal{V}}$ is arbitrary and $\widetilde{\mathcal{V}}$ is dense in $\widetilde{L}^2(\Omega_\rho)$, we have

$$v_{\rho x} = u_{\rho y} \qquad \text{on } \Omega_\rho\,. \tag{6.11}$$

Take any $y_0 \in [\rho, l-\rho]$ and define

$$p_\rho(x,y) = \int_{(0,y_0)}^{(x,y)} u_\rho\, dx + v_\rho\, dy\,. \tag{6.12}$$

Then we have

$$\mathbf{w}_\rho = (u_\rho, v_\rho) = \nabla p_\rho \qquad \text{on} \quad \Omega_\rho\,. \tag{6.13}$$

It is well known that for any fixed interior subdomain Ω' of Ω, $\mathbf{w}_\rho$ converges to $\mathbf{w}$ in $\widetilde{\mathbf{L}}^2(\Omega')$ and then p_ρ converges to a function p in $H^1(\Omega')$ and

$$\mathbf{w} = \nabla p \qquad \text{on } \Omega'\,. \tag{6.14}$$

Since Ω' is arbitrary, we have

$$\mathbf{w} = \nabla p \qquad \text{on} \quad \Omega\,. \tag{6.15}$$

Finally, we show that p is periodic in the x–direction. Let $\mathbf{z}(x,y) = (\psi(y), 0)$, where $\psi \in C_0^\infty([0,l])$. Clearly $\mathbf{z} \in \widetilde{\mathbf{V}}$, and

$$0 = \int_0^l \int_0^1 \mathbf{w}_\rho \cdot \mathbf{z}\, dxdy = \int_0^l \int_0^1 u_\rho(x,y)\, \psi(y)\; dxdy\,. \tag{6.16}$$

Since ψ is from a dense subset of L^2, we obtain

$$\int_0^1 u_\rho(x,y)\; dx = 0 \qquad \text{for} \tag{6.17}$$

With this and with definition (6.12) we obtain that p_ρ, and hence p is periodic in the x–direction.

□

Lemma 6.2. *The norm $\|\mathbf{w}\|_{\widetilde{\mathbf{V}}}$ on $\widetilde{\mathbf{V}}$ is equivalent to the norm $\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^1}$ induced by $\widetilde{\mathbf{H}}^1$.*

Proof. Using the identity

$$u\left(x,y\right)=u\left(x,0\right)+\int_0^y u_y\left(x,y\right)\,dy\,, \tag{6.18}$$

we have

$$\int_0^l\int_0^1 u^2\,dxdy\le 2l\int_0^1 u^2\left(x,0\right)\,dx+l^2\int_0^l\int_0^1 u_y^2\,dxdy\,. \tag{6.19}$$

Similarly, we have

$$\int_0^l\int_0^1 v^2\,dxdy\le \frac{l^2}{2}\int_0^l\int_0^1 v_y^2\,dxdy\,. \tag{6.20}$$

It therefore follows that

$$\int_0^l\int_0^1\left(u^2+v^2\right)\,dxdy\le 2l\int_0^1 u^2\left(x,0\right)\,dx+l^2\int_0^l\int_0^1\left(u_y^2+v_y^2\right)\,dxdy\,, \tag{6.21}$$

which shows that

$$\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^1}\le c\|\mathbf{w}\|_{\widetilde{\mathbf{V}}}\,. \tag{6.22}$$

On the other hand, using (6.18) again, we deduce that

$$\int_0^1 u^2\left(x,0\right)\,dx\le c\int_0^l\int_0^1\left(u^2+u_y^2\right)\,dxdy\,. \tag{6.23}$$

Similarly, we have

$$\int_0^1 u^2\left(x,l\right)\,dx\le c\int_0^l\int_0^1\left(u^2+u_y^2\right)\,dxdy\,. \tag{6.24}$$

It therefore follows that

$$\|\mathbf{w}\|_{\widetilde{\mathbf{V}}}\le c\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^1}\,. \tag{6.25}$$

□

Lemma 6.3. *The norm $\|A\mathbf{w}\|$ on $D\left(A\right)$ is equivalent to the norm $\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^2}$ induced by $\widetilde{\mathbf{H}}^2$.*

Proof. By the definition of the operator A, we have

$$\left(A\mathbf{w}\,\boldsymbol{\Phi}\right)=\left(\left(\mathbf{w}\,\boldsymbol{\Phi}\right)\right)\,,\qquad\forall\mathbf{w}=\left(u,v\right)\in D\left(A\right)\,,\quad\boldsymbol{\Phi}=\left(\xi,\eta\right)\in\widetilde{\mathbf{V}}\,. \tag{6.26}$$

As in the proof of regularity of solutions of the Stokes equations with homogeneous Dirichlet boundary conditions (see, e.g., [9, Chap.3]), we can readily prove that

$$D\left(A\right)=\left\{\mathbf{w}\in\widetilde{\mathbf{H}}\,:\,A\mathbf{w}\in\widetilde{\mathbf{H}}\right\}\,. \tag{6.27}$$

Hence, by Proposition 9 of [15, p.370], $D\left(A\right)$ is a Banach space when provided with the graph norm

$$\|\mathbf{w}\|_{D(A)}=\left(\|\mathbf{w}\|^2+\|A\mathbf{w}\|^2\right)^{1/2}\,.$$

In addition, $D(A)$ with the norm $\|\cdot\|_{\widetilde{\mathbf{H}}^2}$ is also a Banach space, and the norm $\|\cdot\|_{\widetilde{\mathbf{H}}^2}$ is stronger than $\|\cdot\|_{D(A)}$. By the Banach open mapping theorem (see, e.g., [57, p.49]), these two norms $\|\mathbf{w}\|_{\widetilde{\mathbf{H}}^2}$ and $\|\mathbf{w}\|_{D(A)}$ on $D(A)$ are equivalent. On the other hand, by (6.21), we have

$$\|\mathbf{w}\| \leq c\,\|A\mathbf{w}\| \ . \tag{6.28}$$

Hence, the norm $\|A\mathbf{w}\|$ is equivalent to the norm $\|\mathbf{w}\|_{D(A)}$, and then equivalent to the norm induced by $\widetilde{\mathbf{H}}^2$.

□

The following inequality is a special 2–dimensional extension of a classical inequality (see, e.g. [48])

$$\|\varphi\|_{L^q} \leq \beta\,\|\nabla\varphi\|_{L^m}^{\alpha}\,\|\varphi\|_{L^r}^{1-\alpha}\,, \tag{6.29}$$

which holds for any $\varphi \in \overset{\circ}{W}{}^1_m(\Omega)$, $m \geq 2$, $r \geq 1$, where $\Omega \subset \mathbb{R}^2$, $r \leq q < \infty$,

$$\alpha = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{2} - \frac{1}{m} + \frac{1}{r}\right)^{-1}$$

and

$$\beta = \max\left\{\frac{q}{2}; 1 + (m-1)mr\right\}.$$

Here $\overset{\circ}{W}{}^1_m$ denotes the subspace of $L^m(\Omega)$ functions whose gradient is also in $L^m(\Omega)$ and in which the set $C_0^\infty(\Omega)$ is dense.

Lemma 6.4. *For any rectangular region $\Omega = [0,l] \times [0,k] \subset \mathbb{R}^2$, where $k, l > 0$, and for any $\varphi \in H^1(\Omega)$ and $2 \leq q < \infty$ the following inequality holds:*

$$\|\varphi\|_{L^q} \leq \gamma_1\,\|\varphi\| + \gamma_2\,\|\nabla\varphi\|^{\alpha}\,\|\varphi\|^{1-\alpha}\,, \tag{6.30}$$

where $\alpha = 1 - 2/q$ and γ_1, γ_2 are positive constants depending only on the size of Ω and on q.

Proof. Consider an arbitrary $\varphi \in H^1(\Omega)$ and its extension

$$\widetilde{\varphi}(x,y) = \begin{cases} \varphi(x,y) & \text{if } (x,y) \in [0,l]\times[0,k]\ , \\ \left(1+\frac{x}{l}\right)\varphi(-x,y) & \text{if } (x,y) \in [-l,0]\times[0,k]\ , \\ \left(2-\frac{x}{l}\right)\varphi(2l-x,y) & \text{if } (x,y) \in [l,2l]\times[0,k]\ , \\ \left(1+\frac{y}{k}\right)\varphi(x,-y) & \text{if } (x,y) \in [0,l]\times[-k,0]\ , \\ \left(2-\frac{y}{k}\right)\varphi(x,2k-y) & \text{if } (x,y) \in [0,l]\times[k,2k]\ , \\ \left(1+\frac{x}{l}\right)\left(1+\frac{y}{k}\right)\varphi(-x,-y) & \text{if } (x,y) \in [-l,0]\times[-k,0]\ , \\ \left(1+\frac{x}{l}\right)\left(2-\frac{y}{k}\right)\varphi(-x,2k-y) & \text{if } (x,y) \in [-l,0]\times[k,2k]\ , \\ \left(2-\frac{x}{l}\right)\left(1+\frac{y}{k}\right)\varphi(2l-x,-y) & \text{if } (x,y) \in [l,2l]\times[-k,0]\ , \\ \left(2-\frac{x}{l}\right)\left(2-\frac{y}{k}\right)\varphi(2l-x,2k-y) & \text{if } (x,y) \in [0,l]\times[k,2k]\ , \end{cases} \tag{6.31}$$

Inequality (6.29) applies to $\widetilde{\varphi}$ with $\alpha = 1 - 2/q$ and $r = 2 \leq q < \infty$, since $\widetilde{\varphi} \in H^1\left(\widetilde{\Omega}\right)$ and $\widetilde{\varphi}(x, y) = 0$ for $(x, y) \in \partial\widetilde{\Omega}$, where $\widetilde{\Omega} = [-l, 2l] \times [-k, 2k]$. We have

$$\|\widetilde{\varphi}\|_{L^q(\widetilde{\Omega})} \leq \beta \|\nabla\widetilde{\varphi}\|^{\alpha}_{L^2(\widetilde{\Omega})} \|\widetilde{\varphi}\|^{1-\alpha}_{L^2(\widetilde{\Omega})} . \tag{6.32}$$

We have the following relationships between the norms of $\widetilde{\varphi}$ and φ.

$$\|\varphi\|_{L^q(\Omega)} \leq \|\widetilde{\varphi}\|_{L^q(\widetilde{\Omega})} , \tag{6.33}$$

$$\|\widetilde{\varphi}\|^2_{L^2(\widetilde{\Omega})} \leq 9 \|\varphi\|^2 \tag{6.34}$$

and

$$\|\nabla\widetilde{\varphi}\|^2_{L^2(\widetilde{\Omega})} \leq 17 \|\nabla\varphi\|^2 + 6\left(\frac{2}{l^2} + \frac{2}{k^2}\right) \|\varphi\|^2 . \tag{6.35}$$

Inequality (6.33) and (6.34) are trivial consequences of definition (6.31). In order to see the validity of (6.35) one has to estimate the different pieces of $\nabla\widetilde{\varphi}$. One of these estimates, for example is the following:

$$\begin{aligned}
&\int_l^{2l}\int_k^{2k} \left|\nabla\left(\left(2-\frac{x}{l}\right)\left(2-\frac{y}{k}\right)\varphi(2l-x, 2k-y)\right)\right|^2 dxdy = \\
&\int_l^{2l}\int_k^{2k} \left(\frac{1}{l}\left(2-\frac{y}{k}\right)\varphi(2l-x,2k-y) + \left(2-\frac{x}{l}\right)\left(2-\frac{y}{k}\right)\varphi_x(2l-x,2k-y)\right)^2 dxdy \\
&+ \int_l^{2l}\int_k^{2k} \left(\frac{1}{k}\left(2-\frac{x}{l}\right)\varphi(2l-x,2k-y) + \left(2-\frac{x}{l}\right)\left(2-\frac{y}{k}\right)\varphi_y(2l-x,2k-y)\right)^2 dxdy \\
&\leq \frac{2}{l^2}\|\varphi\|^2 + 2\|\varphi_x\|^2 + \frac{2}{k^2}\|\varphi\|^2 + 2\|\varphi_y\|^2 = \left(\frac{2}{l^2} + \frac{2}{k^2}\right)\|\varphi\|^2 + 2\|\nabla\varphi\|^2 .
\end{aligned} \tag{6.36}$$

Combining inequalities (6.32)–(6.35) we obtain

$$\begin{aligned}
\|\varphi\|_{L^q(\Omega)} &\leq \beta\left(17\|\nabla\varphi\|^2 + 6\left(\frac{2}{l^2}+\frac{2}{k^2}\right)\|\varphi\|^2\right)^{\frac{\alpha}{2}} 9^{\frac{1-\alpha}{2}} \|\varphi\|^{1-\alpha} \\
&\leq \beta\left(17^{\frac{\alpha}{2}}\|\nabla\varphi\|^{\alpha} + \left(\frac{12}{l^2}+\frac{12}{k^2}\right)^{\frac{\alpha}{2}}\|\varphi\|^{\alpha}\right) 9^{\frac{1-\alpha}{2}} \|\varphi\|^{1-\alpha} \\
&= \gamma_1\|\varphi\| + \gamma_2\|\nabla\varphi\|^{\alpha}\|\varphi\|^{1-\alpha} .
\end{aligned} \tag{6.37}$$

□

7 Proof of Theorem

We first establish our a priori stability estimates and then deal with questions of existence, uniqueness and regularity.

Let $\mathbf{w} = (u, v)$. We define the energy $E(\mathbf{w})$ of (3.4)–(3.7) as

$$E(\mathbf{w}) = \|\mathbf{w}\|^2 = \int_0^l \int_0^1 \left(u^2 + v^2\right) dxdy , \tag{7.1}$$

and the high order energy $J(\mathbf{w})$ of (3.4)–(3.7) as

$$J(\mathbf{w}) = \|\mathbf{w}\|_{\tilde{\mathbf{V}}}^2 = \int_0^l \int_0^1 \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) dxdy + \frac{1}{k}\int_0^1 \left(u^2(x,0) + u^2(x,l)\right) dx\,. \quad (7.2)$$

Part 1.

Multiplying the first equation of (3.4) by u and the second equation of (3.4) by v and integrating over Ω by parts, we obtain

$$\begin{aligned}
\dot{E}(\mathbf{w}) &= -2\nu \int_0^l \int_0^1 \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) dxdy - 2\int_0^l \int_0^1 \overline{U}' uv\, dxdy \\
&\quad - \int_0^l u^3\big|_{x=0}^1\, dy - \int_0^1 u^2 v\big|_{y=0}^l\, dx - \int_0^l \overline{U} u^2\big|_{x=0}^1\, dy \\
&\quad - \int_0^l uv^2\big|_{x=0}^1\, dy - \int_0^1 v^3\big|_{y=0}^l\, dx - \int_0^l \overline{U} v^2\big|_{x=0}^1\, dy \\
&\quad -2\int_0^l pu\big|_{x=0}^1\, dy - \int_0^1 pv\big|_{y=0}^l\, dx + 2\nu\int_0^l u_x u\big|_{x=0}^1\, dy \\
&\quad +2\nu\int_0^1 u_y u\big|_{y=0}^l\, dx + 2\nu\int_0^l v_x v\big|_{x=0}^1\, dy + 2\int_0^1 v_y v\big|_{y=0}^l\, dx \\
&= -2\nu \int_0^l \int_0^1 \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) dxdy - 2\int_0^l \int_0^1 \overline{U}' uv\, dxdy \\
&\quad +2\nu\int_0^1 u_y u\big|_{y=0}^l\, dx\,. \qquad (7.3)
\end{aligned}$$

Here we have used the relations

$$u_x(0,y,t) = u_x(1,y,t)\,, \quad u_y(0,y,t) = u_y(1,y,t)\,, \quad \text{and} \quad v_y(0,y,t) = v_y(1,y,t)\,, \quad (7.4)$$

which follow from the periodic conditions (3.5)–(3.6) and the divergence free condition. It therefore follows from (6.21) that

$$\begin{aligned}
\dot{E}(\mathbf{w}) &\le -\frac{2\nu}{l^2}E(\mathbf{w}) + \frac{4\nu}{l}\int_0^1 u^2(x,0,t)\, dx + \frac{la}{2\nu}E(\mathbf{w}) \\
&\quad -\frac{2\nu}{k}\int_0^1 \left(u^2(x,l,t) + u^2(x,0,t)\right) dx \\
&= -\frac{2\nu}{l^2}E(\mathbf{w}) + \frac{la}{2\nu}E(\mathbf{w}) \\
&\quad -\int_0^1 \left(2\nu\left(\frac{1}{k} - \frac{2}{l}\right)u^2(x,0,t) + \frac{2\nu}{k}u^2(x,l,t)\right) dx \\
&\le -\left(\frac{2\nu}{l^2} - \frac{al}{2\nu}\right)E(\mathbf{w})\,. \qquad (7.5)
\end{aligned}$$

This implies (5.3).

Part 2.

By (6.21) and (7.3), we have

$$\begin{aligned}\dot{E}(\mathbf{w}) &\leq -2\nu \int_0^l \int_0^1 \left(u_x^2+u_y^2+v_x^2+v_y^2\right) dxdy + \frac{al}{2\nu} E(\mathbf{w}) \\ &\quad -\frac{2\nu}{k}\int_0^1 \left(u^2(x,l,t)+u^2(x,0,t)\right) dx \\ &\leq -\left(2\nu - \frac{al^3}{2\nu}\right)\int_0^l \int_0^1 \left(u_x^2+u_y^2+v_x^2+v_y^2\right) dxdy \\ &\quad -\int_0^1 \left(\left(\frac{2\nu}{k}-\frac{al^2}{\nu}\right)u^2(x,0,t)+\frac{2\nu}{k}u^2(x,l,t)\right) dx \\ &\leq -cJ(\mathbf{w}) \,, \end{aligned} \tag{7.6}$$

where, by (5.1)

$$c = 2\nu - \frac{al^3}{2\nu} > 0\,. \tag{7.7}$$

Multiplying (7.6) by $e^{\sigma t}$, we obtain

$$\frac{d}{dt}\left(e^{\sigma t}E(\mathbf{w})\right) + ce^{\sigma t}J(\mathbf{w}) \leq \sigma e^{\sigma t}E(\mathbf{w}) \leq \sigma E(\mathbf{w}_0)\, e^{-\sigma t}. \tag{7.8}$$

Integrating from 0 to t gives

$$e^{\sigma t}E(\mathbf{w}(t)) + c\int_0^t e^{\sigma s}J(\mathbf{w}(s))\, ds \leq E(\mathbf{w}_0)\left(2-e^{-\sigma t}\right)\,, \tag{7.9}$$

which implies

$$c\int_0^t e^{\sigma s}J(\mathbf{w}(s))\, ds \leq 2E(\mathbf{w}_0)\,, \qquad \forall t \geq 0\,. \tag{7.10}$$

In order to obtain further estimates on J, we multiply the first equation of (3.4) by Au and the second equation of (3.4) by Av and integrate over Ω by parts. This gives

$$\begin{aligned}\int_0^l \int_0^1 (u_t Au + v_t Av)\, dxdy &= \nu \int_0^l \int_0^1 (\Delta u Au + \Delta v Av)\, dxdy \\ &\quad -\int_0^l \int_0^1 \left(uu_x + vu_y + \overline{U}u_x + \overline{U}'v + p_x\right) Au\, dxdy \\ &\quad -\int_0^l \int_0^1 \left(uv_x + vv_y + \overline{U}v_x + p_y\right) Av\, dxdy\,. \end{aligned} \tag{7.11}$$

Since there exists $\mathbf{z} \in \widetilde{\mathbf{H}}^{\perp}$ such that

$$\Delta \mathbf{w} = \mathcal{P}\Delta\mathbf{w} + \mathbf{z}\,, \tag{7.12}$$

we have (noting that $\int_0^l \int_0^1 \mathbf{w}_t \cdot \mathbf{z} \, dxdy = 0$)

$$\begin{aligned}
\int_0^l \int_0^1 \left(u_t A u + v_t A v\right) \, dxdy &= \int_0^l \int_0^1 \left(u_t \Delta u + v_t \Delta v - \mathbf{w}_t \cdot \mathbf{z}\right) \, dxdy \\
&= \int_0^l \left(u_t u_x + v_t v_x\right)\big|_{x=0}^1 \, dy + \int_0^1 \left(u_t u_y + v_t v_y\right)\big|_{y=0}^l \, dx \\
&\quad - \int_0^l \int_0^1 \left(u_{xt} u_x + u_{yt} u_y + v_{xt} v_x + v_{yt} v_y\right) \, dxdy \\
&= -\frac{1}{2} \dot{J}\left(\mathbf{w}\right) , \qquad (7.13)
\end{aligned}$$

and (noting that $\int_0^l \int_0^1 A\mathbf{w} \cdot \mathbf{z} \, dxdy = 0$)

$$\int_0^l \int_0^1 \left(\Delta u A u + \Delta v A v\right) \, dxdy = \int_0^l \int_0^1 \left\|A\mathbf{w}\right\|^2 \, dxdy \, . \qquad (7.14)$$

Moreover, since $A\mathbf{w} \in \widetilde{\mathbf{H}}$ and $\nabla p \in \widetilde{\mathbf{H}}^{\perp}$, we have

$$\int_0^l \int_0^1 \nabla p \cdot A\mathbf{w} \, dxdy = 0 \, . \qquad (7.15)$$

It therefore follows that

$$\begin{aligned}
\dot{J}\left(\mathbf{w}\right) &= -2\nu \left\|A\mathbf{w}\right\|^2 + 2\int_0^l \int_0^1 \left(\left(u u_x + v u_y\right) A u + \left(u v_x + v v_y\right) A v\right) \, dxdy \\
&\quad + 2\int_0^l \int_0^1 \left(\left(\overline{U} u_x + \overline{U}' v\right) A u + \overline{U} v_x A v\right) \, dxdy \, . \qquad (7.16)
\end{aligned}$$

By Lemma 6.4, Young's inequality and Lemma 6.3, we deduce that (the following c's denoting various positive constants that may vary from line to line and ε being a positive constant that will be chosen small enough later)

$$\begin{aligned}
\int_0^l \int_0^1 u u_x A u \, dxdy &\leq \left\|u\right\|_{L^4} \left\|u_x\right\|_{L^4} \left\|Au\right\| \\
&\leq c\left(\left\|\nabla u\right\|^{1/2} \left\|u\right\|^{1/2} + \left\|u\right\|\right) \left(\left\|\nabla u_x\right\|^{1/2} \left\|u_x\right\|^{1/2} + \left\|u_x\right\|\right) \left\|Au\right\| \\
&\leq c\alpha_1\left(E, J\right) + \varepsilon \left\|A\mathbf{w}\right\|^2 , \qquad (7.17)
\end{aligned}$$

where

$$\alpha_1\left(E, J\right) = E\left(\mathbf{w}\right) J\left(\mathbf{w}\right) + E^2\left(\mathbf{w}\right) J\left(\mathbf{w}\right) + J^{3/2}\left(\mathbf{w}\right) E^{1/2}\left(\mathbf{w}\right) + J^2\left(\mathbf{w}\right) E\left(\mathbf{w}\right) \, . \qquad (7.18)$$

In the same way, we can estimate other integrals and obtain

$$\int_0^l \int_0^1 \left(\left(u u_x + v u_y\right) A u + \left(u v_x + v v_y\right) A v\right) \, dxdy \leq c\alpha_1\left(E, J\right) + \varepsilon \left\|A\mathbf{w}\right\|^2 . \qquad (7.19)$$

Further we have

$$\int_0^l \int_0^1 \left(\left(\overline{U}u_x + \overline{U}'v\right) Au + \overline{U}v_x Av\right) dxdy \leq c\left(\varepsilon\right)\left(J\left(\mathbf{w}\right) + E\left(\mathbf{w}\right)\right) + \varepsilon \left\|A\mathbf{w}\right\|^2 . \qquad (7.20)$$

Taking ε small enough, we deduce that.

$$\dot{J}\left(\mathbf{w}\right) \leq c\left(E\left(\mathbf{w}\right) + J\left(\mathbf{w}\right) + \alpha_1\left(E, J\right)\right) - \nu \left\|A\mathbf{w}\right\|^2 . \qquad (7.21)$$

Hence, using (7.10) and applying Lemma 4.1 of [51] with

$$g = c\left(EJ + J^{1/2}E^{1/2}\right), \qquad h = c\left(J + E + EJ + E^2 J\right), \qquad y = J\,, \qquad (7.22)$$

and

$$C_1 = c\left(E\left(\mathbf{w}_0\right) + E^2\left(\mathbf{w}_0\right)\right)\ , \quad C_2 = c\left(E\left(\mathbf{w}_0\right) + E^2\left(\mathbf{w}_0\right) + E^3\left(\mathbf{w}_0\right)\right)\ , \quad C_3 = cE\left(\mathbf{w}_0\right)\ , \qquad (7.23)$$

we deduce that

$$J\left(\mathbf{w}\left(t\right)\right) \leq \beta_1\left(\mathbf{w}_0\right) e^{-\sigma t}, \qquad \forall t \geq 0\,, \qquad (7.24)$$

where

$$\beta_1\left(\mathbf{w}_0\right) = c\left(E\left(\mathbf{w}_0\right) + E^2\left(\mathbf{w}_0\right) + E^3\left(\mathbf{w}_0\right) + J\left(\mathbf{w}_0\right)\exp\left(c\left(E\left(\mathbf{w}_0\right) + E^2\left(\mathbf{w}_0\right)\right)\right)\right)\ . \qquad (7.25)$$

Since $\tau^i \leq ce^{\tau}$, $e^{\tau} \leq ee^{\tau^2}$ for $\tau \geq 0$ and $i = 0, 1, 2, 3$ and $E\left(\mathbf{w}_0\right) \leq c\|\mathbf{w}_0\|^2_{\widetilde{\mathbf{H}}^1}$, we have

$$\beta_1\left(\mathbf{w}_0\right) \leq c\|\mathbf{w}_0\|^2_{\widetilde{\mathbf{H}}^1} \exp\left(c\|\mathbf{w}_0\|^4_{\widetilde{\mathbf{H}}^1}\right)\ . \qquad (7.26)$$

Hence, by Lemma 6.2 and (7.24), we deduce (5.4).

Part 3.

We differentiate the first equation of (3.4) with respect to t and multiply it by u_t and integrate over Ω. This gives

$$\begin{aligned}\int_0^l \int_0^1 u_{tt}u_t\,dxdy \;&=\; \nu\int_0^l \int_0^1 u_t\Delta u_t\,dxdy - \int_0^l \int_0^1 \left(u_t u_x u_t + u u_{xt} u_t + v_t u_y u_t + v u_{yt} u_t\right) dxdy \\ &\quad - \int_0^l \int_0^1 \left(\overline{U}u_{xt}u_t + \overline{U}'v_t u_t + p_{xt}u_t\right) dxdy\,. \qquad (7.27)\end{aligned}$$

Since

$$\begin{aligned}\int_0^l \int_0^1 u_t\Delta u_t\,dxdy \;&=\; \int_0^1 u_{yt}u_t\big|_{y=0}^l\,dx - \int_0^l \int_0^1 \left(u_{xt}^2 + u_{yt}^2\right) dxdy \\ &=\; -\frac{1}{k}\int_0^1 \left(u_t^2\left(x, 0, t\right) + u_t^2\left(x, l, t\right)\right) dx - \int_0^l \int_0^1 \left(u_{xt}^2 + u_{yt}^2\right) dxdy \qquad (7.28)\end{aligned}$$

$$\begin{aligned}\int_0^l \int_0^1 \left(u u_{xt} u_t + v u_{yt} u_t\right) dxdy \;&=\; \frac{1}{2}\int_0^l u u_t^2\big|_{x=0}^1\,dy + \frac{1}{2}\int_0^1 v u_t^2\big|_{y=0}^l\,dx \\ &\quad -\frac{1}{2}\int_0^l \int_0^1 \left(u_x + v_y\right) u_t^2\,dxdy = 0\,, \qquad (7.29)\end{aligned}$$

$$\int_0^l \int_0^1 u_x u_t^2 \, dxdy = \int_0^l u u_t^2 \big|_{x=0}^1 \, dy - 2 \int_0^l \int_0^1 u u_{xt} u_t \, dxdy = -2 \int_0^l \int_0^1 u u_{xt} u_t \, dxdy \,, \quad (7.30)$$

$$\begin{aligned} \int_0^l \int_0^1 v_t u_y u_t \, dxdy &= \int_0^l v_t u u_t \big|_{y=0}^l \, dy - \int_0^l \int_0^1 \left(u v_{yt} u_t + u v_t u_{yt} \right) \, dxdy \\ &= \int_0^l \int_0^1 \left(u u_{xt} u_t - u v_t u_{yt} \right) \, dxdy \,, \end{aligned} \quad (7.31)$$

$$\int_0^l \int_0^1 \overline{U} u_{xt} u_t \, dxdy = \frac{1}{2} \int_0^l \overline{U} u_t^2 \big|_{x=0}^1 \, dy = 0 \,, \quad (7.32)$$

$$\begin{aligned} \int_0^l \int_0^1 p_{xt} u_t \, dxdy &= \int_0^l p_t u_t \big|_{x=0}^1 \, dy - \int_0^l \int_0^1 p_t u_{xt} \, dxdy \\ &= -\int_0^l \int_0^1 p_t u_{xt} \, dxdy \,, \end{aligned} \quad (7.33)$$

we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_0^l \int_0^1 u_t^2 \, dxdy \right) &= -\frac{\nu}{k} \int_0^1 \left(u_t^2 \left(x, 0, t \right) + u_t^2 \left(x, l, t \right) \right) \, dx - \nu \int_0^l \int_0^1 \left(u_{xt}^2 + u_{yt}^2 \right) \, dxdy \\ &\quad + \int_0^l \int_0^1 \left(u u_{xt} u_t + u v_t u_{yt} \right) \, dxdy - \int_0^l \int_0^1 \overline{U}' v_t u_t \, dxdy \\ &\quad + \int_0^l \int_0^1 p_t u_{xt} \, dxdy \,. \end{aligned} \quad (7.34)$$

Differentiating the second equation of (3.4) with respect to t, multiplying it by v_t and integrating over Ω, we obtain

$$\begin{aligned} \int_0^l \int_0^1 v_{tt} v_t \, dxdy &= \nu \int_0^l \int_0^1 v_t \Delta v_t \, dxdy - \int_0^l \int_0^1 \left(u_t v_x v_t + u v_{xt} v_t + v_t v_y v_t + v v_{yt} v_t \right) \, dxdy \\ &\quad - \int_0^l \int_0^1 \left(\overline{U} v_{xt} v_t + p_{yt} v_t \right) \, dxdy \,. \end{aligned} \quad (7.35)$$

Since

$$\begin{aligned} \int_0^l \int_0^1 v_t \Delta v_t \, dxdy &= \int_0^1 v_{yt} v_t \big|_{y=0}^l \, dx - \int_0^l \int_0^1 \left(v_{xt}^2 + v_{yt}^2 \right) \, dxdy \\ &= -\int_0^l \int_0^1 \left(v_{xt}^2 + v_{yt}^2 \right) \, dxdy \,, \end{aligned} \quad (7.36)$$

$$\begin{aligned} \int_0^l \int_0^1 \left(u v_{xt} v_t + v v_{yt} v_t \right) \, dxdy &= \frac{1}{2} \int_0^l u v_t^2 \big|_{x=0}^1 \, dy + \frac{1}{2} \int_0^1 v v_t^2 \big|_{y=0}^l \, dx - \frac{1}{2} \int_0^l \int_0^1 \left(u_x + v_y \right) v_t^2 \, dxdy \\ &= 0 \,, \end{aligned} \quad (7.37)$$

$$\int_0^l \int_0^1 v_y v_t^2 \, dxdy = \int_0^1 v v_t^2 \big|_{y=0}^l \, dx - 2\int_0^l \int_0^1 v v_{yt} v_t \, dxdy$$
$$= -2\int_0^l \int_0^1 v v_{yt} v_t \, dxdy \,, \tag{7.38}$$

$$\int_0^l \int_0^1 u_t v_x v_t \, dxdy = \int_0^l u_t v v_t \big|_{x=0}^1 \, dy - \int_0^l \int_0^1 \left(v u_{xt} v_t + v u_t v_{xt}\right) \, dxdy$$
$$= \int_0^l \int_0^1 \left(v v_{yt} v_t - v u_t v_{xt}\right) \, dxdy \,, \tag{7.39}$$

$$\int_0^l \int_0^1 \overline{U} v_{xt} v_t \, dxdy = \frac{1}{2} \int_0^l \overline{U} v_t^2 \big|_{x=0}^1 \, dy = 0 \,, \tag{7.40}$$

$$\int_0^l \int_0^1 p_{yt} v_t \, dxdy = \int_0^1 p_t v_t \big|_{y=0}^l \, dx - \int_0^l \int_0^1 p_t v_{yt} \, dxdy$$
$$= -\int_0^l \int_0^1 p_t v_{yt} \, dxdy \,, \tag{7.41}$$

we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^l \int_0^1 v_t^2 \, dxdy\right) = -\nu \int_0^l \int_0^1 \left(v_{xt}^2 + v_{yt}^2\right) \, dxdy + \int_0^l \int_0^1 \left(v v_{yt} v_t + v u_t v_{xt}\right) \, dxdy$$
$$+ \int_0^l \int_0^1 p_t v_{yt} \, dxdy \,. \tag{7.42}$$

It therefore follows from (6.21), (7.34) and (7.42) that

$$\begin{aligned}
\dot{E}\left(\mathbf{w}_t\right) &= -2\nu \int_0^l \int_0^1 \left(u_{xt}^2 + u_{yt}^2 + v_{xt}^2 + v_{yt}^2\right) \, dxdy - \frac{2\nu}{k} \int_0^1 \left(u_t^2\left(x,0,t\right) + u_t^2\left(x,l,t\right)\right) \, dx \\
&\quad +2\int_0^l \int_0^1 \left(u u_{xt} u_t + u v_t u_{yt} + v v_{yt} v_t + v u_t v_{xt}\right) \, dxdy - 2\int_0^l \int_0^1 \overline{U}' v_t u_t \, dxdy \\
&\le -\nu J\left(\mathbf{w}_t\right) + \left(\frac{al}{2\nu} - \nu l^{-2}\right) E\left(\mathbf{w}_t\right) \\
&\quad +\frac{2\nu}{l} \int_0^1 u_t^2\left(x,0,t\right) \, dx - \frac{\nu}{k} \int_0^1 \left(u_t^2\left(x,0,t\right) + u_t^2\left(x,l,t\right)\right) \, dx \\
&\quad +2\int_0^l \int_0^1 \left(u u_{xt} u_t + u v_t u_{yt} + v v_{yt} v_t + v u_t v_{xt}\right) \, dxdy \,.
\end{aligned} \tag{7.43}$$

By Lemma 6.4 and Young's inequality, we deduce that (the following c's denoting various positive constants that may vary from line to line and ε being a positive constant that will be determined later)

$$\begin{aligned}\int_0^l\int_0^1 uu_{xt}u_t\,dxdy &\leq \|u\|_{L^4}\|u_t\|_{L^4}\|u_{xt}\|\\ &\leq c\left(\|\nabla u\|^{1/2}\|u\|^{1/2}+\|u\|\right)\left(\|\nabla u_t\|^{1/2}\|u_t\|^{1/2}+\|u_t\|\right)\|u_{xt}\|\\ &\leq c\left(J^{1/4}(\mathbf{w})\,E^{1/4}(\mathbf{w})+E^{1/2}(\mathbf{w})\right)\\ &\quad\times\left(J^{3/4}(\mathbf{w}_t)\,E^{1/4}(\mathbf{w}_t)+E^{1/2}(\mathbf{w}_t)\,J^{1/2}(\mathbf{w}_t)\right)\\ &\leq c\alpha_2(E,J)\,E(\mathbf{w}_t)+\varepsilon J(\mathbf{w}_t)\ , \end{aligned}\tag{7.44}$$

where

$$\alpha_2(E,J)=E(\mathbf{w})\,J(\mathbf{w})+E^2(\mathbf{w})+J(\mathbf{w})+E(\mathbf{w})\ .\tag{7.45}$$

Similarly, we have

$$\int_0^l\int_0^1 uu_{yt}v_t\,dxdy \leq c\alpha_2(E,J)\,E(\mathbf{w}_t)+\varepsilon J(\mathbf{w}_t)\ ,\tag{7.46}$$

$$\int_0^l\int_0^1 vv_{yt}v_t\,dxdy \leq c\alpha_2(E,J)\,E(\mathbf{w}_t)+\varepsilon J(\mathbf{w}_t)\ ,\tag{7.47}$$

$$\int_0^l\int_0^1 vv_{xt}u_t\,dxdy \leq c\alpha_2(E,J)\,E(\mathbf{w}_t)+\varepsilon J(\mathbf{w}_t)\ .\tag{7.48}$$

It therefore follows from (7.43) that

$$\dot{E}(\mathbf{w}_t)\leq(\varepsilon-\nu)\,J(\mathbf{w}_t)-\sigma E(\mathbf{w}_t)+c\alpha_2(E,J)\,E(\mathbf{w}_t)\tag{7.49}$$

which implies

$$\frac{d}{dt}\left(e^{\sigma t}E(\mathbf{w}_t)\right)\leq c\alpha_2(E,J)\,e^{\sigma t}E(\mathbf{w}_t)\ ,\tag{7.50}$$

where σ is given by (5.2). Therefore, by (7.10) and Gronwall's inequality (see. e.g., [44, p.63]), we deduce that

$$E(\mathbf{w}_t(t))\leq E(\mathbf{w}_t(0))\exp\left(cE(\mathbf{w}_0)\left(E(\mathbf{w}_0)+1\right)\right)e^{-\sigma t},\qquad \forall t\geq 0\,.\tag{7.51}$$

On the other hand, by (7.11), (7.14) and (7.15), we have

$$\begin{aligned}\nu\|A\mathbf{w}\|^2 &= \int_0^l\int_0^1\left(\mathbf{w}_t\cdot A\mathbf{w}+(uu_x+vu_y)\,Au+(uv_x+vv_y)\,Av\right)dxdy\\ &\quad+\int_0^l\int_0^1\left(\left(\overline{U}u_x+\overline{U}'v\right)Au+\overline{U}v_xAv\right)dxdy\,. \end{aligned}\tag{7.52}$$

Using (7.19) and (7.20) we obtain

$$\nu\|A\mathbf{w}\|^2\leq c\left(E(\mathbf{w}_t)+\alpha_3(E,J)\right)+\varepsilon\|A\mathbf{w}\|^2\ ,\tag{7.53}$$

where

$$\alpha_3(E,J)=E(\mathbf{w})+J(\mathbf{w})+\alpha_1(E,J)\ .\tag{7.54}$$

Hence, by (5.3), (7.24) and (7.51), we deduce that

$$\|A\mathbf{w}\|^2 \leq \beta_2(\mathbf{w}_0) e^{-\sigma t}, \qquad \forall t \geq 0\,, \tag{7.55}$$

where

$$\beta_2(\mathbf{w}_0) = c\left(E(\mathbf{w}_t(0)) + \sum_{i=1}^{7} E^i(\mathbf{w}_0) + \sum_{i=1}^{4} J^i(\mathbf{w}_0)\right)\exp\left(cE(\mathbf{w}(0))(E(\mathbf{w}(0))+1)\right)\,. \tag{7.56}$$

In addition, multiplying (3.4) by $\mathbf{w}_t$, as in the proof of (7.53), we can prove that

$$E(\mathbf{w}_t) \leq c\left(\|\Delta\mathbf{w}\|^2 + \alpha_3(E,J)\right) + \varepsilon E(\mathbf{w}_t)\,, \tag{7.57}$$

which implies that

$$E(\mathbf{w}_t(0)) \leq c\left(\|\mathbf{w}_0\|^2_{\widetilde{\mathbf{H}}^2} + \sum_{i=1}^{4}\left(E^i(\mathbf{w}_0) + J^i(\mathbf{w}_0)\right)\right)\,. \tag{7.58}$$

Thus, as in (7.26), we deduce that

$$\begin{aligned}\beta_2(\mathbf{w}_0) &\leq c\left(\|\mathbf{w}_0\|^2_{\widetilde{\mathbf{H}}^2} + \sum_{i=1}^{7} E^i(\mathbf{w}_0) + \sum_{i=1}^{4} J^i(\mathbf{w}_0)\right)\exp\left(cE(\mathbf{w}_0)(E(\mathbf{w}_0)+1)\right)\,.\\ &\leq c\|\mathbf{w}_0\|^2_{\widetilde{\mathbf{H}}^2}\exp\left(c\|\mathbf{w}_0\|^2_{\widetilde{\mathbf{H}}^2}\right)\,.\end{aligned} \tag{7.59}$$

Hence, by (7.55) and Lemma 6.3, we deduce (5.5) and inequalities (7.51) and (7.58) show the stated bound of $\|\mathbf{w}_t(t)\|$.

Multiplying the first equation of (3.4) by p_x and the second equation of (3.4) by p_y, integrating over Ω and using (7.19) and (7.20) with $A\mathrm{w}$ replaced by ∇p, we obtain

$$\begin{aligned}\|\nabla p(t)\|^2 &= -\int_0^l\int_0^1 \left(\mathbf{w}_t\cdot\nabla p - \nu\Delta\mathbf{w}\cdot\nabla p + (uu_x+vu_y)p_x + (uv_x+vv_y)\right)dxdy\\ &\quad -\int_0^l\int_0^1\left(\left(\overline{U}u_x + \overline{U}'v\right)p_x + \overline{U}v_xp_y\right)dxdy\\ &\leq c\left(E(\mathbf{w}_t) + \|\Delta\mathbf{w}\|^2 + \alpha_3(E,J)\right) + \varepsilon\|\nabla p\|^2\,.\end{aligned} \tag{7.60}$$

From this last inequality the stated bound on $\|\nabla p\|$ follows by (5.3), (5.4) and (5.5).

Existence and regularity.

We use the Galerkin method to prove existence of solutions. We look for an approximate solution in the form

$$\mathbf{w}^n(x,y,t) = \sum_{i=1}^{n} c_{in}(t)\,\Phi_i(x,y)\,, \qquad n = 1,\ldots \tag{7.61}$$

where the set $\{\Phi_i\}_{i\geq 1}$ forms a Riesz basis in $D(A)$. We require that $\mathbf{w}^n$ satisfies (4.13), i.e.

$$\begin{aligned}&\int_0^l\int_0^1\left(\mathbf{w}_t^n\cdot\mathbf{\Phi}_i + \nu\mathrm{Tr}\{\nabla\mathbf{w}^n\nabla\mathbf{\Phi}_i\} + (\mathbf{w}^n\cdot\nabla)\mathbf{w}^n\cdot\mathbf{\Phi}_i + \overline{U}\mathbf{w}_x^n\mathbf{\Phi}_i + \overline{U}'v^n\xi_i\right)dxdy\\ &= -\frac{\nu}{k}\int_0^1\left(u^n(x,l,t)\,\xi_i(x,l) + u^n(x,o,t)\,\xi_i(x,0)\right)dx\end{aligned} \tag{7.62}$$

for all $\mathbf{\Phi}_i = (\xi_i, \eta_i)$, $i = 1, \ldots, n$. Expanding the definition of $\mathbf{w}^n$, equation (7.62) provides us with a system of first order ordinary differential equations for the time dependent coefficients $\{c_{in}(t)\}_{i\geq 1}$, where we choose the set of initial conditions

$$c_{in}(0) = \int_0^l \int_0^1 \mathbf{w}_0(x,y) \cdot \mathbf{\Phi}_i(x,y)\; dxdy \qquad ß = 1, \ldots, n\,. \tag{7.63}$$

This system depends on $\{c_{in}\}_{i\geq 1}$ analytically, hence, in order to show the existence of a unique solution for all $t \in [0,T]$, it is sufficient to verify the boundedness of $\{|c_{in}(t)|\}_{i\geq 1}$. This is equivalent to the boundedness of the norms $\{\|\mathbf{w}^n(t)\|\}_{n\geq 1}$ as a consequence of the system $\{\mathbf{\Phi}_i\}_{i\geq 1}$ being a Riesz basis. Replacing $\mathbf{\Phi}_i$ by $\mathbf{w}^n$ in (7.62) we deduce estimates (5.3) and (7.10) for $\mathbf{w}^n$. Namely

$$\|\mathbf{w}^n(t)\| \leq \|\mathbf{w}_0^n\|\, e^{-\sigma t} \leq \|\mathbf{w}_0\|\, e^{-\sigma t}, \tag{7.64}$$

and

$$\int_0^T e^{\sigma t} \|\mathbf{w}^n(t)\|^2_{\widetilde{\mathbf{V}}}\, dt \leq M\,\|\mathbf{w}_0^n\| \leq M\,\|\mathbf{w}_0\| \tag{7.65}$$

for some constants M and σ and for a.a. $t \in [0,T]$. In these calculations the steps are justified using the regularity of $\mathbf{w}^n$.

The next step in Galerkin's method is to show that a subsequence of approximating solutions $\{\mathbf{w}^n\}_{n\geq 1}$ converges to a limiting function $\mathbf{w}$ as $n \to \infty$. The convergence is obtained using compactness arguments. In our case, by the uniform boundedness of the sequence $\{\mathbf{w}^n\}_{n\geq 1}$ in $L^2\left([0,T]\,;\widetilde{\mathbf{V}}\right) \cap L^\infty\left([0,T]\,;\widetilde{\mathbf{H}}\right)$ a subsequence $\{\mathbf{w}^n\}_{n\geq 1}$ converges to some element $\mathbf{w} \in L^2\left([0,T]\,;\widetilde{\mathbf{V}}\right) \cap L^\infty\left([0,T]\,;\widetilde{\mathbf{H}}\right)$. The convergence is weak in $L^2\left([0,T]\,;\widetilde{\mathbf{V}}\right)$, weak–star in $L^\infty\left([0,T]\,;\widetilde{\mathbf{H}}\right)$ and, due to compactness ([61, pp. 285–287],) strong in $L^2\left([0,T]\,;\widetilde{\mathbf{V}}\right)$. These convergence properties enable us to prove, as a final step of Galerkin's method, that the limiting function $\mathbf{w}$ is in fact a weak solution of (4.13). We have to show that each term of equation

$$\begin{aligned}
&\frac{d}{dt}\int_0^l \int_0^1 \mathbf{w}^n \cdot \mathbf{\Phi}\, dxdy + \nu \int_0^l \int_0^1 \mathrm{Tr}\left\{\nabla \mathbf{w}^{nT} \nabla \mathbf{\Phi}\right\}\, dxdy + \int_0^l \int_0^1 (\mathbf{w}^n \cdot \nabla)\, \mathbf{w}^n \cdot \mathbf{\Phi}\, dxdy \\
&+ \int_0^l \int_0^1 \overline{U} \mathbf{w}_x^n \mathbf{\Phi}\, dxdy + \int_0^l \int_0^1 \overline{U}' v^n \xi\, dxdy \\
&= -\frac{\nu}{k}\int_0^1 (u^n(x,l,t)\,\xi(x,l) + u^n(x,0,t)\,\xi(x,0))dx
\end{aligned} \tag{7.66}$$

converges to the corresponding term of equation

$$\begin{aligned}
&\frac{d}{dt}\int_0^l \int_0^1 \mathbf{w} \cdot \mathbf{\Phi}\, dxdy + \nu \int_0^l \int_0^1 \mathrm{Tr}\left\{\nabla \mathbf{w}^{T} \nabla \mathbf{\Phi}\right\}\, dxdy + \int_0^l \int_0^1 (\mathbf{w} \cdot \nabla)\, \mathbf{w} \cdot \mathbf{\Phi}\, dxdy \\
&+ \int_0^l \int_0^1 \overline{U} \mathbf{w}_x \mathbf{\Phi}\, dxdy + \int_0^l \int_0^1 \overline{U}' v \xi\, dxdy \\
&= -\frac{\nu}{k}\int_0^1 \left(u(x,l,t)\,\xi(x,l) + u(x,0,t)\,\xi(x,0)\right)\, dx
\end{aligned} \tag{7.67}$$

for all $\mathbf{\Phi} = (\xi, \eta) \in \widetilde{\mathbf{V}}$. This is a standard step in the theory of Navier–Stokes equations for all the terms except the ones on the right hand side of equations (7.66) and (7.67). These terms are present due to our special boundary conditions (3.7). We prove here the convergence of the first term on the right. The convergence of the second term can be proved in the same way. We have to show that

$$\int_0^1 u^n(x,l,t)\,\xi(x,l)\;dx \xrightarrow{n\to\infty} \int_0^1 u(x,l,t)\,\xi(x,l)\;dx \tag{7.68}$$

for all $\mathbf{\Phi} = (\xi, \eta) \in \widetilde{\mathbf{V}}$. We take the difference of the two sides in (7.68) and take the $L^2[0,T]$–inner product of the result by a function $c(t) \in L^2(0,T)$. We obtain

$$\begin{aligned}
&\int_0^T \left(\int_0^1 u^n(x,l,t)\,c(t)\,\xi(x,l)\;dx - \int_0^1 u(x,l,t)\,c(t)\,\xi(x,l)\;dx \right) dt \\
&\le \|\xi\|_{L^\infty} \int_0^T c(t) \int_0^1 \sup_{y\in(0,l)} |u^n(x,y,t) - u(x,y,t)|\;dx\,dt \\
&\le M\,\|\xi\|_{L^\infty} \int_0^T c(t) \int_0^1 \left(\int_0^l |u^n - u|^2\;dy \right)^{1/2} dx\,dt \\
&\quad + M\,\|\xi\|_{L^\infty} \int_0^T c(t) \int_0^1 \left(\int_0^l \left|u^n_y - u_y\right|^2\;dy \right)^{1/4} \left(\int_0^l |u^n - u|^2\;dy \right)^{1/4} dx\,dt\,,
\end{aligned} \tag{7.69}$$

where we used the one–dimensional equivalent of inequality (6.30). We further estimate expressions from (7.69)

$$\begin{aligned}
&\int_0^T c(t) \int_0^1 \left(\int_0^l |u^n - u|^2\;dy \right)^{1/2} dx\,dt \le \int_0^T c(t) \left(\int_0^l \int_0^1 |u^n - u|^2\;dxdy \right)^{1/2} dt \\
&\le \left(\int_0^T c^2(t)\;dt \right)^{1/2} \left(\int_0^T \|\mathbf{w}^n - \mathbf{w}\|^2\,dt \right)^{1/2}.
\end{aligned} \tag{7.70}$$

Here $\int_0^T \|\mathbf{w}^n - \mathbf{w}\|^2\;dt$ converges to zero as $n \to \infty$ according to the strong convergence in $L^2\left([0,T]\,;\widetilde{\mathbf{H}}\right)$. The last expression in (7.69) can be estimated the following way:

$$\begin{aligned}
&\int_0^T c(t) \int_0^1 \left(\int_0^l \left|u^n_y - u_y\right|^2\;dy \right)^{1/4} \left(\int_0^l |u^n - u|^2\;dy \right)^{1/4} dx\,dt \\
&\le \int_0^T c(t)\,\|\nabla(\mathbf{w}^n - \mathbf{w})\|^{1/2}\,\|\mathbf{w}^n - \mathbf{w}\|^{1/2}\,dt \\
&\le \sup_{t\in[0,T]} \left(\|\mathbf{w}^n\|_{\widetilde{\mathbf{V}}} + \|\mathbf{w}\|_{\widetilde{\mathbf{V}}}\right)^{1/2} \left(\int_0^T c^2(t)\;dt \right)^{1/2} \left(\int_0^T \|\mathbf{w}^n - \mathbf{w}\|\;dt \right)^{1/2} \\
&\le \sup_{t\in[0,T]} \left(\|\mathbf{w}^n\|_{\widetilde{\mathbf{V}}} + \|\mathbf{w}\|_{\widetilde{\mathbf{V}}}\right)^{1/2} \left(\int_0^T c^2(t)\;dt \right)^{1/2} \sqrt{T} \left(\int_0^T \|\mathbf{w}^n - \mathbf{w}\|^2\,dt \right)^{1/4}.
\end{aligned} \tag{7.71}$$

Here the last factor converges to zero while the other factors are bounded as $n \to \infty$. Since $c(t) \in L^2(0,T)$ was arbitrary, we obtain the desired convergence result.

It follows from the Helmholtz decomposition (6.4)–(6.5) that, once the existence of weak solutions $\mathbf{w}$ is established, we obtain the existence of pressure p, so that (3.4)–(3.7) are satisfied in a distributional sense.

The rest of the regularity statements in Theorem 5.1 follows from estimates (7.24), (5.4), (7.51), (7.55), (5.5) and from embedding theorems.

Continuous dependence on initial data and uniqueness.

Let $\mathbf{w}_1 = (u_1, v_1)^T$, and $\mathbf{w}_2 = (u_2, v_2)^T$, p_2 be two solutions of (3.4)–(3.7) corresponding to initial data $\mathbf{w}_1^0$ and $\mathbf{w}_2^0$ respectively. Their difference $\mathbf{w} = (u, v)^T = \mathbf{w}_1 - \mathbf{w}_2$, $p = p_1 - p_2$ satisfies

$$u_t - \nu\Delta u + u_1 u_x + u u_{2x} + v_1 u_y + v u_{2y} + \overline{U} u_x + \overline{U}' v + p_x = 0\,, \tag{7.72}$$

$$v_t - \nu\Delta v + u_1 v_x + u v_{2x} + v_1 v_y + v v_{2y} + \overline{U} v_x + p_y = 0\,, \tag{7.73}$$

$$u_x + v_y = 0\,, \tag{7.74}$$

with boundary condition (3.5)–(3.7). Taking the scalar product of (7.72) with u we obtain

$$\begin{aligned}
&\int_0^l \int_0^1 u_t u\, dxdy - \nu \int_0^l \int_0^1 \Delta u u\, dxdy + \int_0^l \int_0^1 u_1 u_x u\, dxdy \\
&+ \int_0^l \int_0^1 u u_{2x} u\, dxdy + \int_0^l \int_0^1 v_1 u_y u\, dxdy + \int_0^l \int_0^1 v u_{2y} u\, dxdy \\
&+ \int_0^l \int_0^1 \overline{U} u_x u\, dxdy + \int_0^l \int_0^1 \overline{U}' v u\, dxdy + \int_0^l \int_0^1 p_x u\, dxdy = 0\,.
\end{aligned} \tag{7.75}$$

Here

$$\begin{aligned}
\int_0^l \int_0^1 u_1 u_x u\, dxdy &= \frac{1}{2} \int_0^l u_1 u^2 \big|_{x=0}^1\, dy - \frac{1}{2} \int_0^l \int_0^1 u_{1x} u^2\, dxdy \\
&\le \frac{1}{2} \|\nabla \mathbf{w}_1\| \, \|\mathbf{w}\|_{L^4}^2 \\
&\le M \|\nabla \mathbf{w}_1\| \left(\|\mathbf{w}\| + \|\nabla \mathbf{w}\|^{1/2} \|\mathbf{w}\|^{1/2} \right)^2 \\
&= M \|\nabla \mathbf{w}_1\| \, \|\mathbf{w}\|^2 + M \|\nabla \mathbf{w}\|^{1/2} \|\nabla \mathbf{w}_1\| \, \|\mathbf{w}\|^{3/2} + M \|\nabla \mathbf{w}\| \, \|\nabla \mathbf{w}_1\| \, \|\mathbf{w}\| \\
&\le M \|\nabla \mathbf{w}_1\| \, \|\mathbf{w}\|^2 + \frac{\delta}{2} \|\nabla \mathbf{w}\|^2 + M \|\nabla \mathbf{w}_1\|^{4/3} \|\mathbf{w}\|^2 + \frac{\delta}{2} \|\nabla \mathbf{w}\|^2 \\
&\quad + M \|\nabla \mathbf{w}_1\|^2 \|\mathbf{w}\|^2 \\
&\le \delta \|\nabla \mathbf{w}\|^2 + M \left(\|\nabla \mathbf{w}_m\| \right) \|\mathbf{w}\|^2\,,
\end{aligned} \tag{7.76}$$

where we used Young's inequality twice in the fourth step with $\delta > 0$ arbitrary and

$$M\left(\|\nabla \mathbf{w}_m(t)\|\right) \equiv c \max_{i=1,2} \left(\|\nabla \mathbf{w}_i\| + \|\nabla \mathbf{w}_i\|^{4/3} + \|\mathbf{w}_i(t)\|^2 \right). \tag{7.77}$$

Terms 4, 5 and 6 in (7.75) can be estimated the same way. The rest of the terms are estimated as in obtaining (5.3). Taking the scalar product of (7.73) with v we obtain

$$\begin{aligned}
&\int_0^l \int_0^1 v_t v\, dxdy - \nu \int_0^l \int_0^1 \Delta v v\, dxdy + \int_0^l \int_0^1 u_1 v_x v\, dxdy + \int_0^l \int_0^1 u v_{2x} v\, dxdy \\
&+ \int_0^l \int_0^1 v_1 v_y v\, dxdy + \int_0^l \int_0^1 v v_{2y} v\, dxdy + \int_0^l \int_0^1 \overline{U} v_x v\, dxdy + \int_0^l \int_0^1 p_y v\, dxdy = 0\,.
\end{aligned} \tag{7.78}$$

The estimation of the terms is similar to (7.75). We obtain from (7.75) and (7.78), after choosing appropriate δ,

$$\frac{d}{dt}\left\|\mathbf{w}\left(t\right)\right\|^{2} \leq M\left(\left\|\nabla \mathbf{w}_{m}\left(t\right)\right\|\right)\left\|\mathbf{w}\left(t\right)\right\|^{2}, . \tag{7.79}$$

Gronwall's inequality applied to (7.79) implies that

$$\left\|\mathbf{w}\left(t\right)\right\|^{2} \leq \left\|\mathbf{w}\left(0\right)\right\|^{2} \exp\left(\int_{0}^{T} M\left(\left\|\nabla \mathbf{w}_{m}\left(\tau\right)\right\|\right) d\tau\right) \tag{7.80}$$

for all $t \in [0, T]$. Since $M\left(\left\|\nabla \mathbf{w}_{m}\left(t\right)\right\|\right)$ is integrable over every finite interval $[0, T]$, (7.80) proves the continuous dependence of solutions on the initial data in the L^2 norm.

8 Numerical Simulation

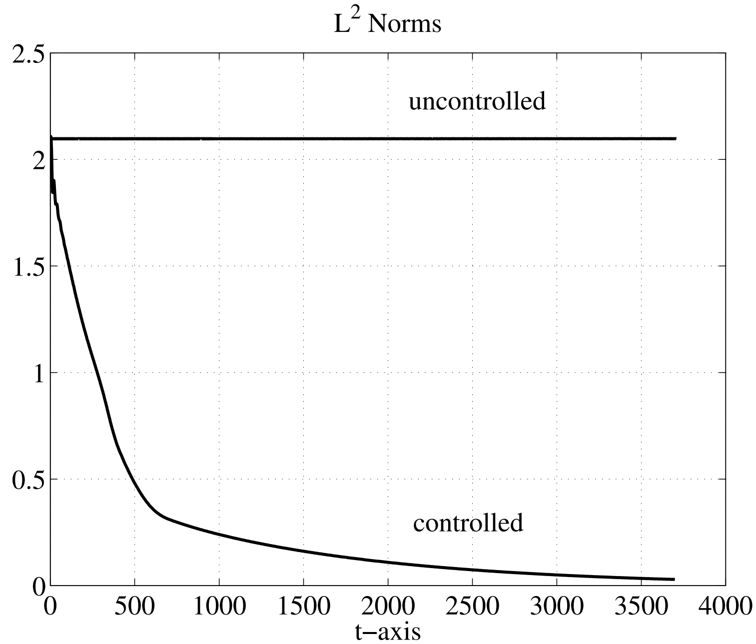

Figure 8.1: Energy Comparison

The simulation example in this section is performed in a channel of length 4π and height 2 for Reynolds number $Re = 15000$ ($a = 2/15000$, $\nu = 1/15000$), which is five orders of magnitude greater than required in Theorem 5.1, and is three times the critical value (5772, corresponding to loss of linear stability) for 2D channel flow. The validity of the stabilization result beyond the assumptions of Theorem 5.1 is not completely surprising since our Lyapunov analysis is based on conservative energy estimates.[2] The control gain used is $k = 1$.

A hybrid Fourier pseudospectral–finite difference discretization and the fractional step technique based on a hybrid Runge–Kutta/Crank–Nicolson time discretization was used to generate the

[2]The effect of boundary control law (3.7) can be seen mathematically in inequality (7.5) in the context of the L^2

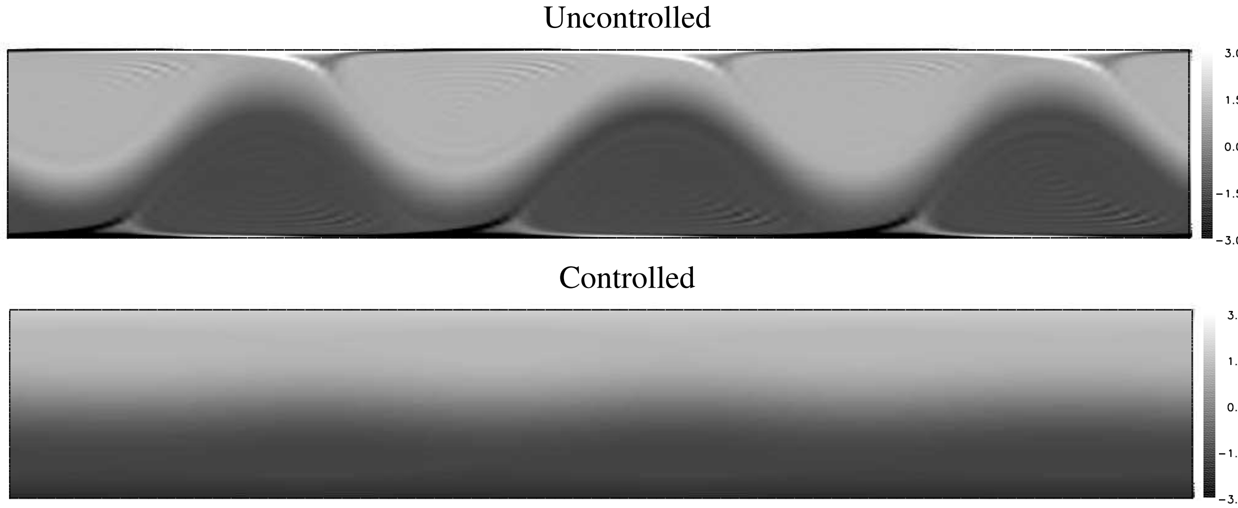

Figure 8.2: Vorticity Maps at $t = 700$.

results. The code originally has been adapted from a Fourier–Chebyshev pseudospectral code of T. Bewley [6], changing the wall-normal discretization to second–order finite differences (P. Blossey, private communication). The nonlinear terms in the Navier–Stokes equations are integrated explicitly using a fourth–order, low storage Runge–Kutta method first devised by Carpenter and Kennedy [8]. The viscous terms are treated implicitly using the Crank–Nicolson method. The numerical method uses "constant volume flux per unit span" instead of the "constant average pressure gradient" assumption to speed up computations. The differences between the two cases are discussed in, for example, [56]. The number of grid points used in our computations was 128×120 and the (adaptive) time step was in the range of $0.05 - 0.07$. The grid points had hyperbolic tangent ($y_j = 1 + \tanh\left(s\left(2\frac{j}{NY} - 1\right)\right) / \tanh(s) \quad j = 0, \ldots, NY$) distribution with stretching factor $s = 1.75$ in the vertical direction in order to achieve high resolution in the critical boundary layer. In order to obtain the flow at the walls in the controlled case the quadratic Three–Point Endpoint Formula was used to approximate the derivatives at the boundary $(U_y(x, 0, t), U_y(x, 2, t))$. This formula is applied in a semi-implicit way in order to avoid numerical instabilities. Namely, the Three–Point Endpoint Formula at the bottom wall has the form

$$U_y(0) \approx d_0 U_0 + d_1 U_1 + d_2 U_2 , \tag{8.2}$$

with notation $U_j = U(y_j)$, $j = 0, 1, 2$ and with appropriate constants d_0, d_1 and d_2. We can write

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perturbation energy. The boundary integral

$$\int_0^1 \left(2\nu\left(\frac{2}{l} - \frac{1}{k}\right) u^2(x, 0, t) - 2\nu\frac{1}{k}u^2(x, l, t)\right) dx \tag{8.1}$$

is negative even for large Reynolds numbers (small kinematic viscosity) if k is sufficiently small. Hence, it improves the stability properties in general. The trace theorem however does not allow us to compare this term and the total energy and to prove the stability results of Theorem 5.1 for large Reynolds numbers. This shows the need for numerical simulation.

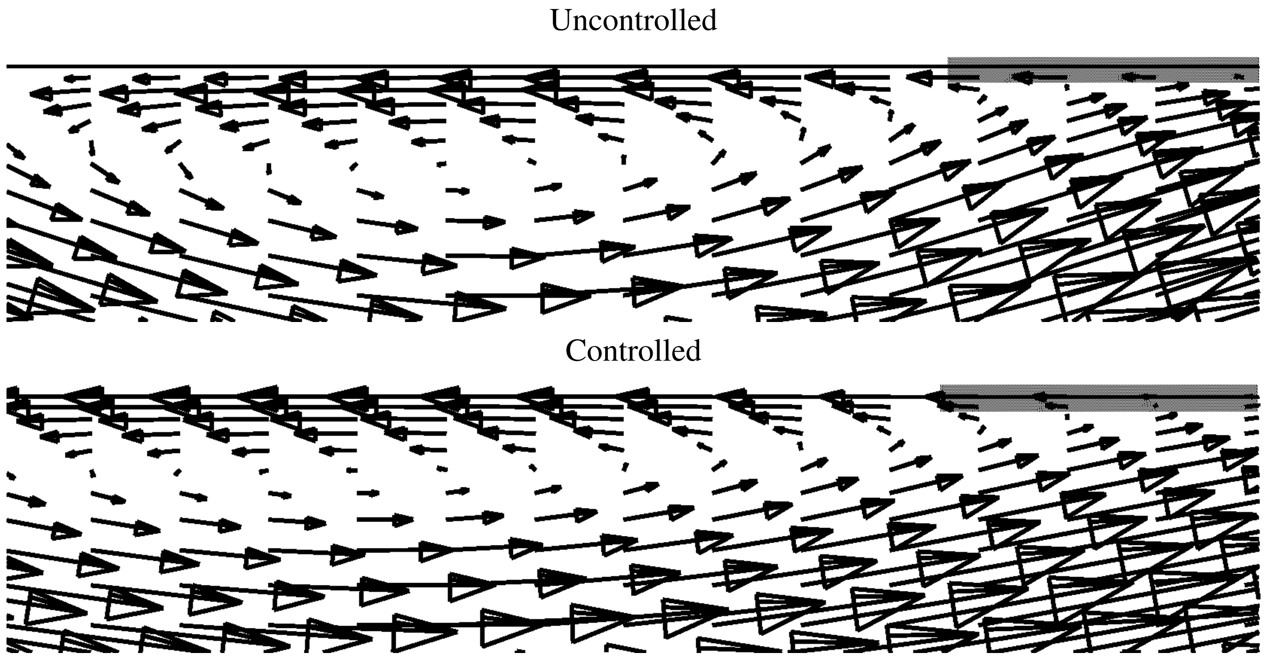

Figure 8.3: Recirculation in the flow at $t = 120$, in a rectangle of dimension 1.37×0.31 zoomed out of a channel of dimension $4\pi \times 2$. The shaded region (upper right corner) is magnified in Figure 8.4.

control law (3.8) now as

$$U_0^{n+1} = k\left[d_0 U_0^{n+1} + d_1 U_1^n + d_2 U_2^n - \frac{al}{2\nu}\right], \tag{8.3}$$

where superscripts n and $n+1$ refer to values at time step n and $n+1$ respectively. Equation (8.3) results in the update law

$$U_0^{n+1} = k\left(d_1 U_1^n + d_2 U_2^n - \frac{al}{2\nu}\right) / (1 - kd_0) \tag{8.4}$$

at the boundary. The boundary condition at the top wall is updated in a similar way. The numerical results show very good agreement with results obtained from a finite volume code used at early stages of simulations. As initial data we consider a statistically steady state flow field obtained from a random perturbation of the parabolic profile over a large time period using the uncontrolled system.

Figure 8.1 shows that our controller achieves stabilization. This is expressed in terms of the L^2–norm of the error between the steady state and the actual velocity field, the so called perturbation energy, which corresponds to system (3.4)–(3.7) with $k = 0$ (zero Dirichlet boundary conditions on the walls) in the uncontrolled case. The initially fast perturbation energy decay somewhat slows down for larger time. What we see here is an interesting example of interaction between linear and nonlinear behavior in a dynamical system. Initially, when the velocity perturbations are large, and the flow is highly nonlinear (exhibiting Tollmien–Schlichting waves with recirculation, see the

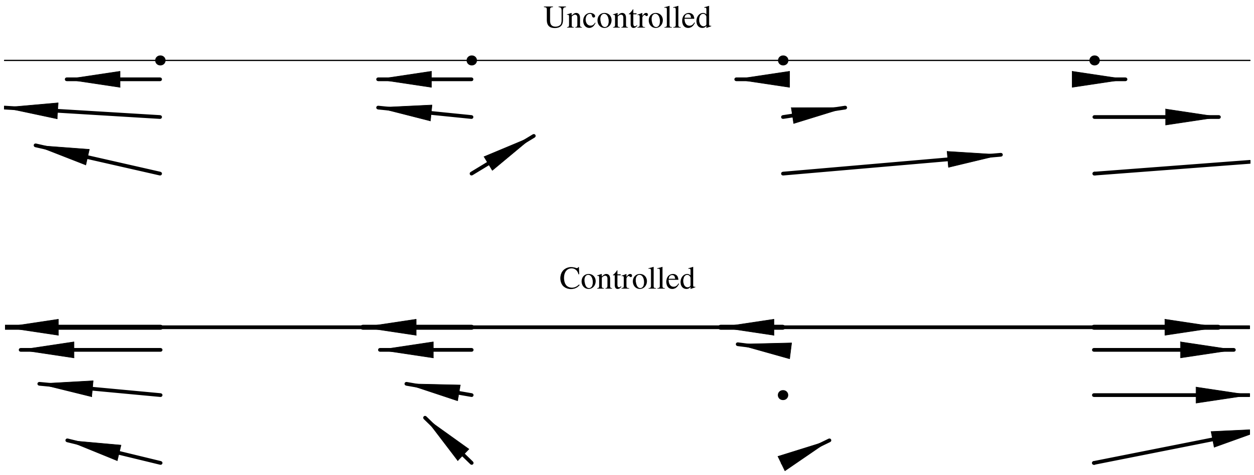

Figure 8.4: Velocity field in a rectangle of dimension 0.393×0.012 zoomed out of a channel of dimension $4\pi \times 2$, at time $t = 120$. The control (thick arrows) acts both *downstream* and *upstream*. The control maintains the value of shear near the desired (laminar) steady–state value.

uncontrolled flow in Figures 8.2 and 8.3). The strong convective (quadratic) nonlinearity dominates over the linear dynamics and the energy decay is fast. Later, at about $t = 500$, the recirculation disappears, the controlled flow becomes close to laminar, and linear behavior dominates, along with its exponential energy decay (with small decay rate).

In the vorticity map, depicted in Figure 8.2 it is striking how uniform the vorticity field becomes for the controlled case, while we observe quasi–periodic bursting (cf. [37]) in the uncontrolled case. We obtained similar vorticity maps of the uncontrolled flow for other (lower) Reynolds numbers, that show agreement qualitatively with the vorticity maps obtained by Jiménez [37]. His paper explains the generation of vortex blobs at the wall along with their ejection into the channel and their final dissipation by viscosity in the uncontrolled case.

The uniformity of the wall shear stress ($U_y|_{\text{wall}}$) in the controlled flow can be also observed in Figure 8.4. Our boundary feedback control (tangential actuation) adjusts the flow field near the upper boundary such that the controlled wall shear stress almost matches that of the steady state profile. The region is at the edge of a small recirculation bubble (Figure 8.3) of the uncontrolled flow, hence there are some flow vectors pointing in the upstream direction while others are oriented downstream. The time is relatively short ($t = 120$) after the introduction of the control and the region is small. As a result it is still possible to see actuation both downstream and upstream. Nevertheless the controlled velocity varies continuously. Figure 8.3 shows that the effect of control is to smear the vortical structures out in the streamwise direction. It is well known that in wall bounded turbulence instabilities are generated at the wall. In two dimensional flows these instabilities are also confined to the walls. As a result, our control effectively stabilizes the flow.

We obtain approximately 71% drag reduction (see Figure 8.5) as a byproduct of our special control law. The drag in the controlled case "undershoots" bellow the level corresponding to the laminar flow and eventually agrees with it up to two decimal places. It is striking that even though

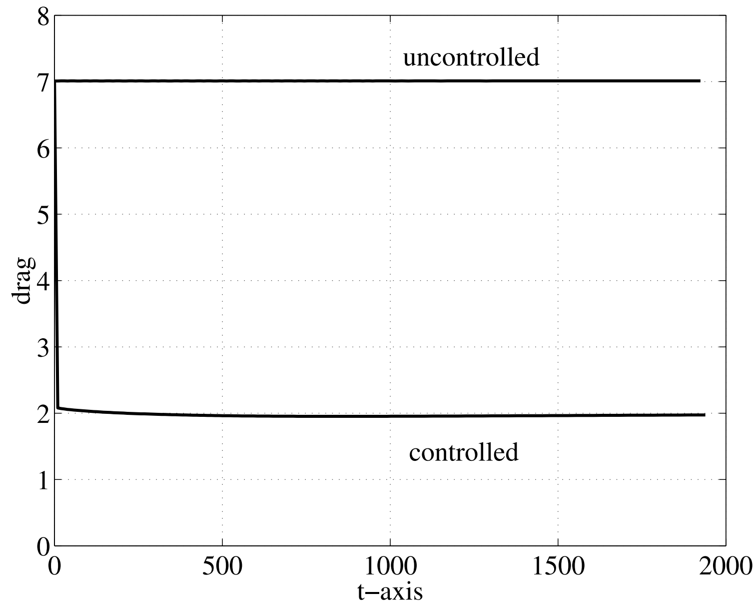

Figure 8.5: Instantaneous drag.

drag reduction was not an explicit control objective (as in most of the works in this field), the stabilization objective results in a controller that reacts to the wall shear stress error, and leads to an almost instantaneous reduction of drag to the laminar level.

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References

[1] R. Adams, *Sobolev Spaces,* Academic Press, New York, 1975.

[2] V. Barbu, "The time optimal control of Navier–Stokes equations," *Systems Control Lett.,* vol. 30, no. 2–3, pp. 93–100, 1997.

[3] V. Barbu and S.S. Sritharan, "H^∞–control theory of fluid dynamics," *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.,* vol. 454, no. 1979, pp. 3009–3033, 1998.

[4] T.R. Bewley, "New frontiers for control in fluid mechanics: a Renaissance approach," *ASME FEDSM 99–6926,* 1999.

[5] T.R. Bewley and S. Liu, "Optimal and robust control and estimation of linear paths to transition," *J. Fluid Mech,* vol. 365, pp. 305–349, 1998.

[6] T.R. Bewley, P. Moin and R. Temam, “DNS–based predictive control of turbulence: an optimal benchmark for feedback algorithms,” submitted to *J. Fluid Mech.,* 2000.

[7] T.R. Bewley, R. Temam and M. Ziane, “A general framework for robust control in fluid mechanics,” *Physica D, vol.* 138, pp. 360–392, 2000.

[8] M.H. Carpenter and C.A. Kennedy, “Fourth-order 2N Runge-Kutta schemes,” *NASA technical memorandum,* no. 109112, 1994.

[9] P. Constantin and C. Foias, *Navier–Stokes Equations,* The University of Chicago Press, Chicago and London, 1988.

[10] J.-M. Coron, “On the controllability of 2–D incompressible perfect fluids,” *J. Math. Pures Appl.* vol. 75 pp. 155–188, 1996

[11] J.-M. Coron, On the controllability of the 2–D incompressible “Navier–Stokes equations with the Navier slip boundary conditions,” *ESAIM: Control, Optim. Cal. Var.,* vol. 1, pp. 35–75, 1996.

[12] J.-M.Coron, On null asymptotic stabilization of the 2–D Euler equation of incompressible fluids on simply connected domains,” *Prepublications 98–59* Université de Paris–Sud, Mathematiques, 1998.

[13] J.-M. Coron and A.V. Fursikov, “Global exact controllability of the 2–D Navier–Stokes equations on a manifold without boundary,” *Russian J. Math. Phys.,* vol. 4, no. 4, pp. 429–448, 1996.

[14] L. Cortelezzi, J.L. Speyer, K.H. Lee and K. Kim, “Robust reduced–order control of turbulent channel flows via distributed sensors and actuators,” *Proc. of the 37th IEEE CDC,* Tampa, FL, December 1998, pp. 1906–1911.

[15] R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology, Vol.2, Functional and variational methods,* Springer–Verlag, Berlin, 1992.

[16] M. Desai and K. Ito, “Optimal controls of Navier–Stokes equations,” *SIAM J. Control Optim.,* vol. 32, no. 5, pp. 1428–1446, 1994.

[17] C. Fabre, “Uniqueness results for Stokes equations and their consequences in linear and nonlinear control problems,” *ESAIM: Control, Optim. Cal. Var.*, vol. 1, pp. 267–302, 1996.

[18] H.O. Fattorini and S.S. Sritharan, “Existence of optimal controls for viscous flow problems,” *Proceedings of the Royal Society of London, Series A,* vol. 439, pp. 81–102, 1992.

[19] H.O. Fattorini and S.S. Sritharan, “Necessary and sufficient conditions for optimal controls in viscous flow problems,” *Proceedings of the Royal Society of Edinburgh, Series A,* vol. 124A, pp. 211–251, 1994.

[20] H.O. Fattorini and S.S. Sritharan, “Optimal chattering controls for viscous flow,” *Nonlinear Anal.,* vol. 25, no. 8, pp. 763–797, 1995.

[21] H.O. Fattorini and S.S. Sritharan, "Optimal control problems with state constraints in fluid mechanics and combustion," *Appl. Math. Optim.,* vol. 38, no. 2, pp. 159–192, 1998.

[22] E. Fernández–Cara, "On the approximate and null controllability of the Navier–Stokes equations," *SIAM Rev.,* vol. 41, no. 2, pp. 269–277, 1999.

[23] E. Fernández–Cara and M. González–Burgos, "A result concerning controllability for the Navier–Stokes equations," *SIAM J. Control Optim.,* vol. 33, no. 4, pp. 1061–1070, 1995.

[24] E. Fernández–Cara and J. Real, "On a conjecture due to J.L. Lions," *Nonlinear Anal., TMA,* vol. 21, pp. 835–847, 1993.

[25] A.V. Fursikov, "Exact boundary zero controllability of three–dimensional Navier–Stokes equations," *J. Dynam. Control Systems,* vol. 1, no. 3, pp. 325–350, 1995.

[26] A.V. Fursikov, M.D. Gunzburger and L.S. Hou, "Boundary value problems and optimal boundary control for the Navier–Stokes system: the two–dimensional case." *SIAM J. Control Optim.,* vol. 36, no. 3, pp. 852–894, 1998.

[27] A.V. Fursikov and O.Y. Imanuvilov, "On exact boundary zero–controllability of two–dimensional Navier–Stokes equations, Mathematical problems for Navier–Stokes equations (Centro, 1993)," *Acta Appl. Math.,* vol. 37, no. 1–2, pp. 67–76, 1994.

[28] A.V. Fursikov and O.Y. Imanuvilov, "Local exact boundary controllability of the Navier–Stokes system," *Optimization methods in partial differential equations (South Hadley, MA, 1996), 115–129, Contemp. Math.,* vol. 209, Amer. Math. Soc., Providence, RI, 1997.

[29] A.V. Fursikov and O.Y. Imanuvilov, "Local exact controllability of the Navier–Stokes equations," *C. R. Acad. Sci. Paris Ser. I Math.,* vol. 323, no. 3, pp. 275–280, 1996.

[30] M.D. Gunzburger, L.S. Hou, S. Manservisi and Y. Yan, "Computations of optimal controls for incompressible flows. Flow control and optimization." *Int. J. Comput. Fluid Dyn.,* vol. 11, no. 1–2, pp. 181–191, 1998.

[31] M.D. Gunzburger, L.S. Hou, T.P. Svobodny, "Boundary velocity control of incompressible flow with an application to viscous drag reduction," *SIAM J. Control Optim.,* vol. 30, no. 1, pp. 167–181, 1992.

[32] C. He, "Weighted estimates for nonstationary Navier–Stokes equations," *J. Differential Equations,* vol. 148, pp. 422–444, 1998.

[33] L.S Hou and Y. Yan, "Dynamics and approximations of a velocity tracking problem for the Navier–Stokes flows with piecewise distributed control," *SIAM J. Control Optim.,* vol. 35, no. 6, pp. 1847–1885, 1997 .

[34] L.S Hou and Y. Yan, "Dynamics for controlled Navier–Stokes systems with distributed controls," *SIAM J. Control Optim.,* vol. 35, no. 2, pp. 654–677, 1997.

[35] O.Y. Imanuvilov, "On exact controllability for the Navier–Stokes equations," *ESAIM: Control, Optim. Cal. Var.*, vol. 3, pp. 97–131, 1998.

[36] K. Ito and S. Kang, "A dissipative feedback control synthesis for systems arising in fluid dynamics," *SIAM J. Control Optim.,* vol. 32, no. 3, pp. 831–854, 1994.

[37] J. Jiménez, "Transition to turbulence in two–dimensional Poiseuille flow," *J. Fluid Mech.,* vol. 218, pp. 265–297, 1990.

[38] S.S. Joshi, J.L. Speyer and J. Kim, "A system theory approach to the feedback stabilization of infinitesimal and finite–amplitude disturbances in plane Poiseuille flow," *Journal of Fluid Mechanics,* vol. 332, pp. 157–184, 1997.

[39] S.S. Joshi, J.L. Speyer and J. Kim, "Finite dimensional optimal control of Poiseuille flow," *Journal of Guidance, Control, and Dynamics,* vol. 22, no. 2, pp. 340–348, 1999.

[40] R. Kajikiya and T. Miyakawa, "On L^2 decay of weak solutions of the Navier–Stokes equations in R^n," *Math. Z.,* vol. 192, pp. 135–148, 1986.

[41] T. Kato, "Strong L^p–solutions of the Navier–Stokes equations in R^m, with application to weak solutions," *Math. Z.,* vol. 187, pp. 471–480, 1984.

[42] T. Kawanago, "Stability estimate for strong solutions of the Navier–Stokes system and its applications," *Electronic J. Differential Equations*, vol. 1998, no. 15, pp. 1–23, 1998.

[43] L.R. Keefe, "Method and apparatus for reducing the drag of flows over surfaces," US Patent US5803409, 1998.

[44] H.K. Khalil, *Nonlinear Systems*, Prentice–Hall, Inc., New Jersey, 1996.

[45] V. Komornik, D.L. Russell and B.-Y. Zhang, "Stabilisation de l'équation de Korteweg–de Vries," *C. R. Acad. Sci. Paris Sér. I Math.,* vol. 312, no. 11, pp. 841–843, 1991.

[46] H. Kozono, "Global L^n–solution and its decay property for the Navier–Stokes equations in half–space R^n_+," *J. Differential Equations,* vol. 79, pp. 79–88, 1989.

[47] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Second English edition, Gordon and Breach, Science Publisher, Inc., New York, 1969.

[48] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasi–linear Equations of Parabolic Type,* Translations of AMS, vol. 23, 1968.

[49] J.L. Lions and E. Magenes, *Non–homogeneous Boundary value Problems and Applications, Vol.I,* Springer–Verlag, Berlin, Heidelberg, New York, 1972.

[50] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I,* Clarendon Press, Oxford, 1996.

[51] W.J. Liu and M. Krstić, "Stability enhancement by boundary control in the Kuramoto–Sivashinsky equation," to appear in *Nonlinear Analysis, TMA,* 1999.

[52] K. Masuda, "Weak solutions of Navier–Stokes equations," *Tohoku Math. Journ.*, vol. 36, pp. 623–646, 1984.

[53] C. Qu and P. Wang, "L^p Exponential stability for the equilibrium solutions of the Navier–Stokes equations," *J. Math. Anal. Appl.,* vol. 190, pp. 419–427, 1995.

[54] L. Rosier, "Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain," *ESAIM Control Optim. Calc. Var.,* vol. 2, pp. 33–55, 1997. (electronic)

[55] D.L. Russell and B.-Y. Zhang, "Exact controllability and stabilizability of the Korteweg–de Vries equation," *Trans. Amer. Math. Soc.,* vol. 348, no. 9, pp. 3643–3672, 1996.

[56] B.L. Rozhdestvensky and I.N. Simakin, "Secondary flows in a plane channel: their relationship and comparison with turbulent flows," *J. Fluid Mech.*, vol. 147, pp. 261–289, 1984.

[57] W. Rudin, *Functional Analysis*, McGraw–Hill Book Company, New York, 1973.

[58] M. Schonbek, "L^2 decay for weak solutions of the Navier–Stokes equations," *Arch. Rational Mech. Anal.,* vol. 88, pp. 209–222, 1985.

[59] B.L. Smith and A. Glezer, "The formation and evolution of synthetic jets," *Phys. Fluids,* vol. 10, no. 9, pp. 2281–2297, 1998.

[60] S.S. Sritharan, "Dynamic programming of the Navier–Stokes equations," *Systems Control Lett.,* vol. 16, no. 4, pp. 299–307, 1991.

[61] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis,* Third (Revised) edition, North–Holland Publishing Company, Amsterdam, 1984.

[62] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis,* Second edition, SIAM, Philadelphia, 1995.

[63] M. Wiegner, "Decay and stability in L_p for strong solutions of the Cauchy problem for the Navier–Stokes equations," *Numerical Functional Analysis and Optimization,* vol. 18, pp. 143–188, 1997.

[64] M. Wiegner, "Decay results for solutions of the Navier–Stokes equations on R^n," *J. London Math. Soc.*, vol. 35, pp. 303–313, 1987.

[65] L. Zhang, "Sharp rate of decay of solutions to 2–dimensional Navier–Stokes equations," *Comm. Partial Differential Equations,* vol. 20, no. 1&2, pp. 119–127, 1995.