Backstepping in Infinite Dimension for a Class of Parabolic Distributed Parameter Systems *

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Abstract

In this paper a family of stabilizing boundary feedback control laws for a class of linear parabolic PDEs motivated by engineering applications is presented. The design procedure presented here can handle systems with an arbitrary finite number of open-loop unstable eigenvalues and is not restricted to a particular type of boundary actuation. The stabilization is achieved through the design of coordinate transformations that have the form of recursive relationships. The fundamental difficulty of such transformations is that the recursion has an infinite number of iterations. The problem of feedback gains growing unbounded as grid becomes infinitely fine is resolved by a proper choice of the target system to which the original system is transformed. We show how to design coordinate transformations such that they are sufficiently regular (not continuous but L_∞). We then establish closed–loop stability, regularity of control, and regularity of solutions of the PDE. The result is accompanied by a simulation study for a linearization of a tubular chemical reactor around an unstable steady state.

Keywords: boundary control, linear parabolic PDEs, stabilization, backstepping, coordinate transformations.

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1 Introduction

Motivated by the model for the chemical tubular reactor, the model of unstable burning in solid rocket propellants, and other PDE systems that appear in various engineering applications, we present an algorithm for global stabilization of a broader class of linear parabolic PDEs. The result presented here is a generalization of the ideas of Balogh and Krstić [BK1]. The goal is to obtain an L_∞ coordinate transformation and a boundary control law that renders the closed–loop system asymptotically stable, and additionally establish regularity of control and regularity of solutions for the closed–loop system.

The key issue with arbitrarily unstable linear parabolic PDE systems is the target system to which one is transforming the original system by coordinate transformation. For example, if one takes the standard backstepping route leading to a tri–diagonal form, the resulting transformations, if thought of as integral transformations, end up with "kernels" that are not even finite. A proper selection of the target system will result in a bounded kernel and the solutions corresponding to the controlled problem are going to be at least continuous.

The class of parabolic PDEs considered in this paper is

$$u_t(t,x) = \varepsilon u_{xx}(t,x) + Bu_x(t,x) + \lambda(x)\, u(t,x) + \int_0^x f(x,\xi)u(t,\xi)d\xi, \quad x \in (0,1), \quad t > 0, \tag{1.1}$$

where $\varepsilon > 0$ and B are constants, $\lambda(x) \in L_\infty(0,1)$ and $f(x,y) \in L_\infty([0,1]\times[0,1])$, with initial condition $u(0,x) = u^0(x)$, for $x \in [0,1]$. The boundary condition at $x = 0$ is either homogeneous Dirichlet,

$$u(t,0) = 0, \qquad t > 0, \tag{1.2}$$

or homogeneous Neumann,

$$u_x(t,0) = 0, \qquad t > 0. \tag{1.3}$$

while the Dirichlet boundary condition (alternatively Neumann) at the other end

$$u(t,1) = \alpha(u(t))^*, \qquad t > 0 \tag{1.4}$$

is used as the control input, where the linear operator α represents a control law to be designed to achieve stabilization. It is assumed that the initial distribution is compatible with (1.2) (alternatively with (1.3)), i.e. $u^0(0) = 0$ (alternatively $u^0_x(0) = 0$).

Our interest in systems described by (1.1) is twofold. First, the physical motivation for considering equation (1.1) is that it represents the linearization of the class of reaction–diffusion–convection equations that model many physical phenomena. Examples are numerous and among others include the problem of compressor rotating stall (the most recent model due to Mezic [HMBZ] is $u_t = \varepsilon u_{xx} + u - u^3$), whose linearization is (1.1) with $\lambda(x) \equiv 1$, $B \equiv f(x,y) \equiv 0$, the unstable heat equation [BK3] ($\varepsilon \equiv 1$, $B \equiv f(x,y) \equiv 0$, and $\lambda(x) \equiv \lambda$ = constant), the linearization of the unstable burning for solid rocket propellants [BK5] ($\varepsilon \equiv B \equiv 1$, $\lambda(x) \equiv 0$, and $f(x,y) = -Ae^{-x}\delta(y)$, A = constant), and the linearization of an adiabatic chemical tubular reactor around either stable or unstable equilibrium [HH1] $\left(\varepsilon = \frac{1}{Pe},\ B = -1,\ \lambda(x)\right.$ spatially dependent function that corresponds to either stable or unstable steady state profile, and $\left. f(x,y) \equiv 0\right)$.

Second, from the perspective of control theory, systems described by (1.1) are interesting since their discretization appears in the most general strict-feedback form [KKK]. Therefore, developing backstepping control algorithms for such a class of problems is of great importance as the first step in an attempt to fully extend the existing backstepping techniques from the finite dimensional setup to the infinite dimensional one.

*Throughout the paper we use the simplified notation $u(t) = u(t,.)$.

For different combinations of the boundary condition at $x = 0$ (Dirichlet or Neumann), and control applied at $x = 1$ (Dirichlet or Neumann), we use a backstepping method for the finite difference semi–discretized approximation of (1.1) to derive a boundary feedback control law that makes the infinite dimensional closed loop system stable with an arbitrary prescribed stability margin. We show that the integral kernel in the control law resides in the function space $L_\infty(0,1)$ and that solutions corresponding to the controlled problem are classical.

We should stress that although we focus our attention in this paper on a class of 1D parabolic problems, the design procedures and results presented here can be easily extended to higher–dimensional problems. We have demonstrated that fact in [BK4], where backstepping was successfully applied on a 2D nonlinear heat convection model from Burns *et al.* [BKR2]. Note that a further extension to 3D would be conceptually the same and the control would be applied via a planar array of wall actuators and the coordinate transforms in the backstepping design would depend on three indices (α_{ijk},β_{ijk},γ_{ijk}). As already mentioned, the main issue in our approach is not the dimension of the system, but the choice of the target system that will result in a bounded kernel as grid becomes infinitely fine.

The prior work on stabilization of general parabolic equations includes, among others, the results of Triggiani [T1] and Lasiecka and Triggiani [LT1] who developed a general framework for the structural assignment of eigenvalues in parabolic problems through the use of semigroup theory. Separating the open loop system into a finite dimensional unstable part and an infinite dimensional stable part, they apply feedback boundary control that stabilizes the unstable part while leaving the stable part stable. An LQR approach in Lasiecka and Triggiani [LT2] is also applicable to this problem. A unified treatment of both interior and boundary observations/control generalized to semilinear problems can be found in [A2]. Nambu [N] developed auxiliary functional observers to stabilize diffusion equations using boundary observation and feedback. Stabilizability by boundary control in the optimal control setting is discussed by Bensoussan et al. [BDDM]. For the general Pritchard–Salamon class of state–space systems a number of frequency–domain results has been established on stabilization during the last decade (see, e.g. [C] and [L2] for a survey). The placement of finitely many eigenvalues were generalized to the case of moving infinitely many eigenvalues by Russel [R2]. The stabilization problem can be also approached using the abstract theory of boundary control systems developed by Fattorini [F1] that results in a dynamical feedback controller (see remarks in [CZ, Section 3.5]). Extensive surveys on the controllability and stabilizability theory of linear partial differential equations can be found in [R1, LT2].

The first result, to our knowledge, where backstepping was applied to a PDE is the control design for a rotating beam by Coron and d'Andrea–Novel [CA]. They designed a nonlinear feedback torque control law for a hyperbolic PDE model of rotating beam with no damping and no control on the free boundary. The scalar control input, applied in a distributed fashion, is used to achieve global asymptotic stabilization of the system. In addition, authors show regularity of control inputs.

Backstepping was successfully applied to parabolic PDEs in [LK, BK4, BK5, BK2] in settings with only a finite number of steps.

Our work is also related to results of Burns, King and Rubio [BKR1]. Although their result is quite different because of the different control objective (theirs is LQR optimal control, ours is stabilization), and the fact that their plant is open–loop stable but with the spatial domain of dimension higher than ours, the technical problem of proving some regularity of the gain kernel ties the two results together.

In an attempt to generalize the backstepping techniques from finite dimensions to linear parabolic infinite dimensional systems, Boskovic *et al.* [BK3] considered the unstable heat equation with parameters restricted so that the number of open–loop unstable eigenvalues is no greater than one. In this limited case we derived a closed–form and smooth coordinate transformation based on backstepping. In an effort to extend the results from [BK3] for an arbitrary level of instability, Balogh and Krstić [BK1] obtained the first backstepping type feedback control law for a linear PDE that can accommodate for an arbitrary level of instability, i.e. stabilize the system that has an arbitrary number of unstable eigenvalues in open–loop. By designing a sufficiently regular (not continuous but L_∞) coordinate transformation they were able to establish closed–loop stability, regularity of control, and regularity of solutions of the PDE.

We emphasize that, in addition to being an important step in a generalization of a finite dimensional technique to infinite dimensions and with the ultimate goal of potential applications to nonlinear problems, the backstepping control design for linear parabolic PDEs presented here has advantages of its own. First, compared to the pole placement type of designs that have been prevalent in the control of parabolic PDEs, it has the standard advantage of a Lyapunov based approach that the designer does not have to look for the solution of the uncontrolled system to find the controller that stabilizes it. The problem of finding modal data in the case of spatially dependent $\lambda(x)$ and $f(x,y)$ becomes nontrivial and finding closed form expressions for the system eigenvalues and eigenvectors appears highly unlikely in the general case. In some instances, as is the case for the tubular reactor example used in our simulation study, the only way to obtain spatially dependent coefficients is numerically. In that case finding eigenvalues and eigenvectors numerically becomes inevitable, which might be computationally very expensive if a large number of grid points is necessary for simulating the system. To obtain a backstepping controller that stabilizes the system, on the other hand, the designer has to obtain a kernel given by a simple recursive expression that is computationally inexpensive. Second, from applications point of view, numerical results both for the nonlinear [BK4, BK5, BK2] and linear (linearization of the chemical tubular reactor presented here) parabolic PDEs suggest that reduced order backstepping control laws (designed on a much coarser grid) that use only a few state measurements can successfully stabilize the system.

The main reason for choosing a model of a chemical tubular reactor in our simulation study is because a large research activity has been dedicated to the control designs based on PDE models of chemical reactors. Using a combination of Galerkin's method with a procedure for the construction of approximate inertial manifolds, Christofides [C2] designed output feedback controllers for nonisothermal tubular reactors that guarantee stability and enforce the output of the closed–loop system to follow, up to a desired accuracy, a prescribed response for almost all times. In a more recent paper by Orlov and Dochain [OD] a sliding mode control developed for minimum phase semilinear infinite–dimensional systems was applied to stabilization of both plug flow (hyperbolic) and tubular (parabolic) chemical reactors. Both results use distributed control to stabilize the system around prespecified temperature and concentration steady state profiles. On the other hand, we apply point actuation at $x = 1$ in our design.

The paper is organized as follows. In Section 2 we formulate our problem and its discretization for two different cases of boundary conditions at $x = 0$ (either homogeneous Dirichlet $u(t,0) = 0$, or homogeneous Neumann $u_x(t,0) = 0$) and we lay out our strategy for the solution of the stabilization problem. The precise formulations of our main theorems are contained in Section 3. In Lemmas 1 (homogeneous Dirichlet at $x = 0$) and 5 (homogeneous Neumann at $x = 0$) of Section 4 we design coordinate transformations for semi–discretizations of the system (for a less general case with no integral term on the RHS of the system equation) which map them into exponentially stable systems. We show in Lemmas 2 (homogeneous Dirichlet at $x = 0$) and 6 (homogeneous Neumann at $x = 0$) that the discrete coordinate transformations remain uniformly bounded as the grid gets refined and hence converge to coordinate transformations in the infinite dimensional case. The regularity $C_w([0,1], L_\infty(0,1))$ of the transformation is established in Lemma 3. We complete the proofs of our main theorems using Lemma 4 (Balogh and Krstić [BK1]) that establishes the stability of the infinite dimensional controlled systems. The extension from Dirichlet to Neumann type of actuation is presented in Section 5, followed by an extension of the result to the case when the integral term is present on the RHS of the system equation in Section 6. Finally, simulation study for a linearized model of an adiabatic chemical tubular reactor presented in Section 7 shows, besides the effectiveness of our control, that reduced versions of the controller stabilize the infinite dimensional system as well.

2 Motivation

In this section we formulate our problem for a particular case of the system (1.1) with no integral term on the RHS of the system equation, i.e. for

$$u_t(t,x) = \varepsilon u_{xx}(t,x) + Bu_x(t,x) + \lambda(x)u(t,x), \qquad x \in (0,1), \quad t > 0. \tag{2.5}$$

This particular case is the most interesting from the applications point of view and we present results for all four combinations of different types of boundary conditions at the uncontrolled end $x = 0$, and actuations at the control end $x = 1$.

An extension of the result for the most general case of the system (1.1) (integral term on the RHS of the system equation) with homogeneous Dirichlet boundary condition at $x = 0$ and Dirichlet type of actuation at $x = 1$ is presented in Section 6.

2.1 Case 1: Dirichlet boundary condition at $x = 0$

In this subsection we analyze the case when homogeneous Dirichlet boundary condition is imposed at $x = 0$. We first introduce the case when actuation of Dirichlet type is applied at $x = 1$. The extension for the Neumann type of actuation is presented is Section 5. The semi–discretized version of system (2.5) with (1.2) and (1.4) using central differencing in space is the finite dimensional system:

$$u^0 = 0, \tag{2.6}$$

$$\dot{u}_i = \varepsilon\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + B\frac{u_{i+1} - u_i}{h} + \lambda_i u_i, \qquad i = 1,\ldots,n, \tag{2.7}$$

$$u_{n+1} = \alpha_n(u_1, u_2, \ldots, u_n), \tag{2.8}$$

where $n \in N$, $h = \frac{1}{n+1}$ and $u_i = u(t,ih)$, $\lambda_i = \lambda(ih)$, for $i = 0,\ldots,n+1$. With u_{n+1} as control, this system is in the strict–feedback form and hence it is readily stabilizable by standard backstepping. However the naive version of backstepping would result in a control law with gains that grow unbounded as $n \to \infty$.

The problem with standard backstepping is that it would not only attempt to stabilize the equation, but also place all of its poles, and thus as $n \to \infty$, change its parabolic character. Indeed, an infinite–dimensional version of the tridiagonal form in backstepping is not parabolic. Our approach will be to transform the system, but keep its parabolic character, i.e., keep the second spatial derivative in the transformed coordinates.

Towards this end, we start with a finite–dimensional backstepping–style coordinate transformation

$$w_0 = u_0 = 0, \tag{2.9}$$

$$w_i = u_i - \alpha_{i-1}(u_1,\ldots,u_{i-1}), \qquad i = 1,\ldots,n, \tag{2.10}$$

$$w_{n+1} = 0, \tag{2.11}$$

for the discretized system (2.6)–(2.8), and seek the functions α_i such that the transformed system has the form

$$w_0 = 0, \tag{2.12}$$

$$\dot{w}_i = \varepsilon\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + B\frac{w_{i+1} - w_i}{h} - cw_i, \qquad i = 1,\ldots,n, \tag{2.13}$$

$$w_{n+1} = 0. \tag{2.14}$$

The finite–dimensional system (2.12)-(2.14) is the semi–discretized version of the infinite–dimensional system

$$w_t(t,x) = \varepsilon w_{xx}(t,x) + Bw_x(t,x) - cw(t,x), \qquad x \in (0,1), \quad t > 0, \tag{2.15}$$

with boundary conditions

$$w(t,0) = 0, \tag{2.16}$$
$$w(t,1) = 0, \tag{2.17}$$

which is exponentially stable for $c > -\varepsilon\pi^2 - \frac{B^2}{4\varepsilon}$.

The backstepping coordinate transformation is obtained by combining (2.6)–(2.8), (2.9)–(2.11) and (2.12)–(2.14) and solving the resulting system for the α_i's. Namely, subtracting (2.13) from (2.7), expressing the obtained equation in terms of $u_k - w_k$, $k = i-1, i, i+1$, and applying (2.10) we obtain the recursive form

$$\begin{aligned}\alpha_i \;=\; & (\varepsilon + Bh)^{-1}\Bigg\{\left(2\varepsilon + Bh + ch^2\right)\alpha_{i-1} - \varepsilon\alpha_{i-2} - (c+\lambda_i)\,h^2 u_i \\ & + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial u_j}\left((\varepsilon+Bh)\,u_{j+1} - \left(2\varepsilon + Bh - \lambda_j h^2\right)u_j + \varepsilon u_{j-1}\right)\Bigg\},\end{aligned} \tag{2.18}$$

for $i = 1,\ldots,n$ with initial values $\alpha_0 = 0$ and[†]

$$\alpha_1 = -\frac{h^2}{\varepsilon + Bh}(c+\lambda_1)\,u_1. \tag{2.19}$$

Writing the α_i's in the linear form

$$\alpha_i = \sum_{j=1}^{i} k_{i,j}u_j\,, \qquad i = 1,\ldots,n \tag{2.20}$$

and performing simple calculations we obtain the general recursive relationship

$$k_{i,1} = \frac{h^2}{\varepsilon+Bh}(c+\lambda_1)\,k_{i-1,1} + \frac{\varepsilon}{\varepsilon+Bh}\left(k_{i-1,2} - k_{i-2,1}\right), \tag{2.21}$$
$$k_{i,j} = \frac{h^2}{\varepsilon+Bh}(c+\lambda_j)\,k_{i-1,j} + k_{i-1,j-1} + \frac{\varepsilon}{\varepsilon+Bh}\left(k_{i-1,j+1} - k_{i-2,j}\right), \qquad j = 2,\ldots,i-2, \tag{2.22}$$
$$k_{i,i-1} = \frac{h^2}{\varepsilon+Bh}(c+\lambda_{i-1})\,k_{i-1,i-1} + k_{i-1,i-2}\,, \tag{2.23}$$
$$k_{i,i} = k_{i-1,i-1} - \frac{h^2}{\varepsilon+Bh}(c+\lambda_i)\,, \tag{2.24}$$

for $i = 4,\ldots,n$ with initial conditions

$$k_{1,1} = -\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)\,, \tag{2.25}$$
$$k_{2,1} = -\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda_1)^2\,, \tag{2.26}$$
$$k_{2,2} = -\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1) + \frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\right), \tag{2.27}$$
$$k_{3,1} = -\frac{h^6}{(\varepsilon+Bh)^3}(c+\lambda_1)^3 - \frac{\varepsilon}{(\varepsilon+Bh)}\frac{h^2}{(\varepsilon+Bh)}(c+\lambda_2)\,, \tag{2.28}$$
$$k_{3,2} = -\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1) + \frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\right) - \frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda_1)^2\,, \tag{2.29}$$
$$k_{3,3} = -\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1) + \frac{h^2}{\varepsilon+Bh}(c+\lambda_2) + \frac{h^2}{\varepsilon+Bh}(c+\lambda_3)\right). \tag{2.30}$$

[†]From now on we assume that n is large enough to have the inequality $\varepsilon + Bh > 0$ satisfied.

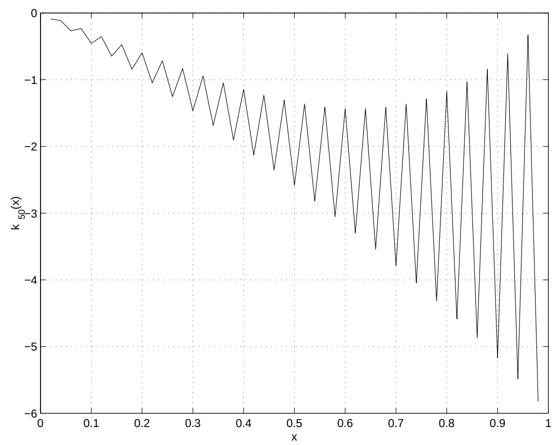

Figure 1: Oscillation of the approximating kernel for $n = 50$, $\lambda = 5$, $\varepsilon = 1$, $B = 1$, $c = 1$.

For the simple case when $\lambda(x) \equiv \lambda$ = constant, equations (2.21)–(2.30) can be solved explicitly to obtain

$$k_{i,i-j} = -\binom{i}{j+1} L_n{}^{j+1} - (i-j) \sum_{l=1}^{[j/2]} \frac{1}{l} \binom{j-l}{l-1} \binom{i-l}{j-2l} L_n{}^{j-2l+1} M_n{}^{l} \tag{2.31}$$

for $i = 1, \ldots, n,\ \ j = 0, \ldots, i-1$, where

$$L_n = \frac{h^2}{\varepsilon + Bh} (c + \lambda) , \tag{2.32}$$

$$M_n = \frac{\varepsilon}{\varepsilon + Bh} . \tag{2.33}$$

Regarding the infinite dimensional system (2.5) with (1.2) and (1.4), the linearity of the control law in (2.20) suggests a stabilizing boundary feedback control of the form

$$\alpha(u) = \int_0^1 k(x) u(x)\, dx, \tag{2.34}$$

where the function $k(x)$ is obtained as a limit of $\left\{(n+1) k_{n,j}\right\}_{j=1}^{n}$ as $n \to \infty$. From the complicated expression (2.31) it is not clear if such limit exists. A quick numerical simulation (see Figure 1) shows that the coefficients $\left\{(n+1) k_{n,j}\right\}_{j=1}^{n}$ remain bounded but it also shows their oscillation, and increasing n only increases the oscillation (see Figure 2). A similar type of behavior was encountered in the related work of Balogh and Krstić [BK1]. Clearly, there is no hope for pointwise convergence to a continuous kernel $k(x)$. However, as we will see in the next sections, there is weak* convergence in L_∞ as we go from the finite dimensional case to the infinite dimensional one. As a result, we obtain a solution to our stabilization problem (2.5) with boundary conditions (1.2) and (1.4).

2.2 Case 2: Neumann boundary condition at $x = 0$

If a homogeneous Neumann boundary condition is prescribed at $x = 0$, a slightly different procedure has to be applied. Note that we may assume without the loss of generality that the boundary condition at $x = 0$ is homogeneous since the boundary condition of the third kind can be easily converted

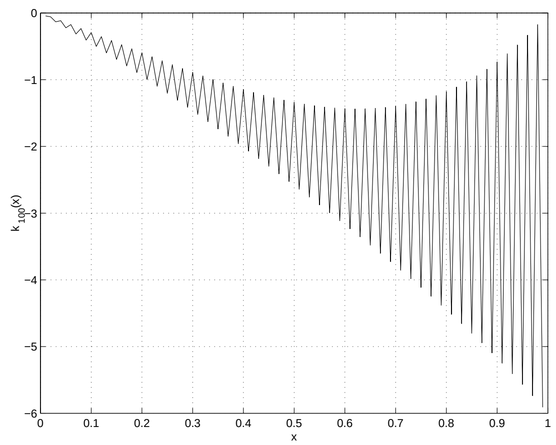

Figure 2: Oscillation of the approximating kernel for $n = 100$, $\lambda = 5$, $\varepsilon = 1$, $B = 1$, $c = 1$.

into a homogeneous Neumann boundary condition without changing the qualitative structure of the system equation. For example, the boundary condition $u_x(0) = qu(0)$, q = constant, would be converted by a variable change $z(t,x) = u(t,x)e^{-qx}$. The system equation for that case would be transformed into $z_t = \varepsilon z_{xx} + (B+2q\varepsilon)z_x + (\lambda(x)+Bq+\varepsilon q^2)z$. The main idea for the case of a homogeneous Neumann boundary condition is very similar to the case with homogeneous Dirichlet boundary condition at $x = 0$, and we only outline the differences.

We start with a finite–dimensional backstepping–style coordinate transformation

$$
\begin{aligned}
w_0 &= u_0\,, && (2.35)\\
w_1 &= u_1\,, && (2.36)\\
w_i &= u_i - \alpha_{i-1}(u_1,\ldots,u_{i-1})\,, \qquad i = 2,\ldots,n\,, && (2.37)\\
w_{n+1} &= 0\,, && (2.38)
\end{aligned}
$$

that transforms the original system into the semi–discretized version of the infinite–dimensional system

$$
w_t(t,x) = \varepsilon w_{xx}(t,x) + Bw_x(t,x) - cw(t,x), \qquad x \in (0,1), \quad t > 0\,, \tag{2.39}
$$

with boundary conditions

$$
\begin{aligned}
w_x(t,0) &= 0\,, && (2.40)\\
w(t,1) &= 0\,, && (2.41)
\end{aligned}
$$

which is exponentially stable for $c > -\varepsilon\pi^2 - \frac{B^2}{4\varepsilon}$. Note that the given bound is not optimal. The optimal bound is $c > -\varepsilon\eta^2 - \frac{B^2}{4\varepsilon}$, where η is the smallest positive root of equation $-\frac{2}{B}\eta = \tan(\eta)$.

Using the same approach as in Subsection 2.1 we obtain

$$
\begin{aligned}
\alpha_i = {}& (\varepsilon+Bh)^{-1}\Bigg\{\left(2\varepsilon+Bh+ch^2\right)\alpha_{i-1} - \varepsilon\alpha_{i-2} - (c+\lambda_i)\,h^2u_i + \frac{\partial\alpha_{i-1}}{\partial u_1}\left((\varepsilon+Bh)\,u_2 - \left(\varepsilon+Bh-\lambda_1h^2\right)u_1\right)\\
&+ \sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial u_j}\left((\varepsilon+Bh)u_{j+1} - \left(2\varepsilon+Bh-\lambda_jh^2\right)u_j + \varepsilon u_{j-1}\right)\Bigg\}\,,
\end{aligned} \tag{2.42}
$$

instead of (2.18), with $\alpha_0 = 0$ and α_1 given by (2.19). Writing the α_i's in the linear form (2.20) we obtain

$$
k_{i,1} = \left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1) + \frac{\varepsilon}{\varepsilon+Bh}\right)k_{i-1,1} + \frac{\varepsilon}{\varepsilon+Bh}\left(k_{i-1,2} - k_{i-2,1}\right), \tag{2.43}
$$

and $k_{i,j}$, $k_{i,i-1}$ and $k_{i,i}$ given by (2.22)–(2.24). The initial conditions for the recursion are given as

$$k_{2,1} = -\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{\varepsilon}{\varepsilon+Bh}\right)\frac{h^2}{\varepsilon+Bh}(c+\lambda_1), \tag{2.44}$$

$$k_{3,1} = -\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{\varepsilon}{\varepsilon+Bh}\right)^2\frac{h^2}{\varepsilon+Bh}(c+\lambda_1) - \frac{\varepsilon}{(\varepsilon+Bh)}\frac{h^2}{(\varepsilon+Bh)}(c+\lambda_2), \tag{2.45}$$

$$k_{3,2} = -\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\right) - \frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda_1)^2-\frac{\varepsilon}{\varepsilon+Bh}\frac{h^2}{\varepsilon+Bh}(c+\lambda_1), \tag{2.46}$$

and $k_{1,1}$, $k_{2,2}$, and $k_{3,3}$ the same as for the Dirichlet case. For the simple case when $\lambda(x)\equiv\lambda$ = constant, equation (2.31) becomes

$$\begin{aligned} k_{i,i-j} = & -\begin{pmatrix} i \\ j+1 \end{pmatrix} {L_n}^{j+1} - (i-j)\sum_{l=1}^{[j/2]}\frac{1}{l}\begin{pmatrix} j-l \\ l-1 \end{pmatrix}\begin{pmatrix} i-l \\ j-2l \end{pmatrix}{L_n}^{j-2l+1}{M_n}^{l} \\ & -\sum_{l=0}^{[(j-1)/2]}\sum_{k=0}^{j-2l-1}\begin{pmatrix} l+k \\ l \end{pmatrix}\begin{pmatrix} i-l-1 \\ k \end{pmatrix}{M_n}^{j-l-k}{L_n}^{k+1} \\ & +\sum_{l=1}^{[(j-1)/2]}\sum_{k=1}^{j-2l-1}\begin{pmatrix} l+k \\ l-1 \end{pmatrix}\begin{pmatrix} i-l-1 \\ k-1 \end{pmatrix}{M_n}^{j-l-k}{L_n}^{k+1}. \end{aligned} \tag{2.47}$$

Same as for the Dirichlet case, the stabilizing boundary feedback control will be in the form (2.34), where the function $k(x)$ is obtained as a limit of $\left\{(n+1)k_{n,j}\right\}_{j=1}^{n}$ for $k_{n,j}$ from (2.47) as $n\to\infty$.

3 Main Result

3.1 Case 1: Dirichlet boundary condition at $x=0$

As we stated earlier, we use a backstepping scheme for the semi–discretized finite difference approximation of system (2.5), (1.2), (1.4), (2.34) to derive a linear boundary feedback control law that makes the infinite dimensional closed loop system stable with an arbitrary prescribed stability margin. The precise formulation of our main result is given by the following theorem.

Theorem 1. *For any $\lambda(x)\in L_\infty(0,1)$, and $\varepsilon, c>0$ there exists a function $k\in L_\infty(0,1)$ such that for any $u^0\in L_\infty(0,1)$ the unique classical solution $u(t,x)\in C^1\left((0,\infty);C^2(0,1)\right)$ of system (2.5), (1.2), (1.4), (2.34) is exponentially stable in the $L_2(0,1)$ and maximum norms with decay rate c. The precise statements of stability properties are the following: There exists positive constant M[‡] such that for all $t>0$*

$$\|u(t)\|_2 \le M\left\|u^0\right\|_2 e^{-ct} \tag{3.48}$$

and

$$\max_{x\in[0,1]}|u(t,x)|\le M\sup_{x\in[0,1]}\left|u^0(x)\right|e^{-ct}. \tag{3.49}$$

Remark 1. For a given integral kernel $k\in L_\infty(0,1)$ the existence and regularity results for the corresponding solution $u(t,x)$ follows from trivial modifications in the proof of [L1, Thm 4.1]. See also [F2].

[‡] M grows with c, λ and $1/\varepsilon$.

3.2 Case 2: Neumann boundary condition at $x = 0$

Theorem 2. *For any $\lambda(x) \in L_\infty(0,1)$, and $\varepsilon, c > 0$ there exists a function $k \in L_\infty(0,1)$ such that for any $u^0 \in L_\infty(0,1)$ the unique classical solution $u(t,x) \in C^1\left((0,\infty); C^2(0,1)\right)$ of system (2.5), (1.3), (1.4), (2.34) is exponentially stable in the $L_2(0,1)$ and maximum norms with decay rate c. The precise statements of stability properties are the following: There exists positive constant M[§] such that for all $t > 0$*

$$\|u(t)\|_2 \le M \left\|u^0\right\|_2 e^{-ct} \tag{3.50}$$

and

$$\max_{x\in[0,1]} |u(t,x)| \le M \sup_{x\in[0,1]} \left|u^0(x)\right| e^{-ct}. \tag{3.51}$$

4 Proof of Main Result

4.1 Case 1: Dirichlet boundary condition at $x = 0$

As it was already mentioned in the introduction, the proof of Theorem 1 requires four lemmas.

Lemma 1. *The elements of the sequence $\left\{k_{i,j}\right\}$ defined in (2.21)–(2.30) satisfy*

$$|k_{i,i-j}| \le \binom{i}{j+1} {L_n}^{j+1} + (i-j) \sum_{l=1}^{[j/2]} \frac{1}{l} \binom{j-l}{l-1} \binom{i-l}{j-2l} {L_n}^{j-2l+1} {M_n}^l \tag{4.52}$$

where $\lambda = \max_{x\in[0,1]} |\lambda(x)|$.

Remark 2. There is equality in (4.52) when $\lambda(x) \equiv \lambda$ = constant > 0.

Proof. The right hand side of equations (2.25)–(2.30) can be estimated to obtain estimates for the initial values of k's

$$|k_{1,1}| \le \frac{h^2}{\varepsilon + Bh}(c+\lambda) = L_n, \tag{4.53}$$

$$|k_{2,1}| \le \frac{h^4}{(\varepsilon + Bh)^2}(c+\lambda)^2 = {L_n}^2, \tag{4.54}$$

$$|k_{2,2}| \le 2\frac{h^2}{\varepsilon + Bh}(c+\lambda) = 2L_n, \tag{4.55}$$

$$|k_{3,1}| \le \frac{h^6}{(\varepsilon + Bh)^3}(c+\lambda)^3 + \frac{\varepsilon}{\varepsilon + Bh}\frac{h^2}{\varepsilon + Bh}(c+\lambda) = {L_n}^3 + M_n L_n, \tag{4.56}$$

$$|k_{3,2}| \le 3\frac{h^4}{(\varepsilon + Bh)^2}(c+\lambda)^2 = 3{L_n}^2, \tag{4.57}$$

$$|k_{3,3}| \le 3\frac{h^2}{\varepsilon + Bh}(c+\lambda) = 3L_n. \tag{4.58}$$

We then go from $j = i$ backwards to obtain from (2.24) and (2.23)

$$|k_{i,i}| \le i\frac{h^2}{\varepsilon + Bh}(c+\lambda) = iL_n, \tag{4.59}$$

$$|k_{i,i-1}| \le \frac{i(i-1)}{2}\frac{h^4}{(\varepsilon + Bh)^2}(c+\lambda)^2 = \frac{i(i-1)}{2}{L_n}^2. \tag{4.60}$$

[§] M grows with c, λ and $1/\varepsilon$.

Finally we obtain inequality (4.52) of Lemma 1 using the general identity (2.22) and mathematical induction. □

In order to prove that the finite dimensional coordinate transformation (2.9), (2.10), (2.20) converges to an infinite dimensional one that is well–defined, we show the uniform boundedness of $(n+1)k_{i,j}$ with respect to $n \in \mathbb{N}$ as $i = 1,\dots,n,\ j = 1,\dots,i$. Note that the binomial coefficients in inequality (4.52) are monotone increasing in i and hence it is enough to show the boundedness of terms $(n+1)k_{n,j}$, or equivalently $(n+1)k_{n,n-j}$. Also, we introduce notations

$$q = \frac{j}{n} \in [0,1]\,, \tag{4.61}$$

and

$$E = 2\frac{\lambda+c}{\varepsilon}\,, \tag{4.62}$$

$$R = \frac{2|B|}{\varepsilon}\,, \tag{4.63}$$

so that we can write

$$\begin{aligned} |k_{n,n-j}| &= |k_{n,n-qn}| \\ &\leq \binom{n}{qn+1} {L_n}^{qn+1} + (n-qn)\sum_{l=1}^{[qn/2]} \frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l} {L_n}^{qn-2l+1} {M_n}^{l}\,, \end{aligned} \tag{4.64}$$

$$L_n = \frac{h^2}{\varepsilon+Bh}(c+\lambda) \leq \frac{E}{(n+1)^2}\,, \tag{4.65}$$

and

$$M_n = \frac{\varepsilon}{\varepsilon+Bh} = 1 - \frac{Bh}{\varepsilon+Bh} \leq 1 + \frac{|B|h}{\frac{\varepsilon}{2}} = 1 + \frac{R}{n+1}\,, \tag{4.66}$$

for sufficiently large n.

Lemma 2. *The sequence* $\left\{(n+1)k_{n,j}\right\}_{j=1,\dots,n,\, n\geq 1}$ *remains bounded uniformly in* n *and* j *as* $n \to \infty$.

Proof. We can write, according to (4.64),

$$\begin{aligned} (n+1)|k_{n,n-qn}| \leq\ & (n+1)\binom{n}{qn+1}\left(\frac{E}{(n+1)^2}\right)^{qn+1} \\ & + (n+1)(n-qn)\sum_{l=1}^{[qn/2]} \frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1} {M_n}^{l}\,. \end{aligned} \tag{4.67}$$

The first term can be estimated as

$$\begin{aligned} (n+1)\binom{n}{qn+1}\left(\frac{E}{(n+1)^2}\right)^{qn+1} &\leq (n+1)^{qn+2}\left(\frac{E}{n+1}\right)^{qn}\frac{E}{(n+1)^{qn+2}} \\ &\leq E\left(\frac{E}{n}\right)^{qn} \\ &\leq Ee^{E/e}, \end{aligned} \tag{4.68}$$

where the last line shows that the bound is uniform in n and also in q.

In the following steps we will use the simple inequalities

$$\begin{aligned}\frac{(n-l)!}{(n-qn+l)!} &\leq \frac{n}{n-qn+2l}\frac{n-1}{n-qn+2l-1}\cdots\frac{n-l+1}{n-qn+l+1}\frac{(n-l)!}{(n-qn+l)!}\\ &= \frac{n!}{(n-qn+2l)!}\end{aligned} \tag{4.69}$$

and

$$\frac{(qn-l)!}{l!\,(qn-2l+1)!}\left(\frac{1}{n+1}\right)^{qn-2l} \leq q \tag{4.70}$$

to obtain

$$\begin{aligned}&(n+1)(n-nq)\sum_{l=1}^{[qn/2]}\frac{1}{l}\begin{pmatrix} qn-l \\ l-1\end{pmatrix}\begin{pmatrix} n-l \\ qn-2l\end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1} {M_n}^l\\ &\leq E\frac{(n+1)\,n}{(n+1)^2}\sum_{l=1}^{[qn/2]}\frac{(qn-l)!}{l!\,(qn-2l+1)!}\left(\frac{1}{n+1}\right)^{qn-2l}\\ &\quad\times\frac{n!}{(qn-2l)!\,(n-qn+2l)!}\left(\frac{E}{n+1}\right)^{qn-2l}\left(1+\frac{R}{n+1}\right)^{l}\\ &\leq Eq\left(1+\frac{R}{n+1}\right)^{nq}\sum_{s=0}^{nq}\begin{pmatrix} n \\ s\end{pmatrix}\left(\frac{E}{n}\right)^{s}1^{n-s}\\ &\leq Eq\left(1+\frac{R}{n}\right)^{nq}\left(1+\frac{E}{n}\right)^{nq}\\ &\leq Ee^{R+E}\,.\end{aligned}$$

Here in the last step we used the fact that the convergence $\left(1+\frac{X}{n}\right)^n \xrightarrow{n\to\infty} e^X$ is monotone increasing and $q\in[0,1]$. This proves the lemma. □

As a result of the above boundedness, we obtain a sequence of piecewise constant functions

$$k_n(x,y) = (n+1)\sum_{i=1}^{n}\sum_{j=1}^{i}k_{i,j}\chi_{I_{i,j}}(x,y)\,, \qquad (x,y)\in[0,1]\times[0,1]\,, \qquad n\geq 1\,, \tag{4.71}$$

where

$$I_{i,j} = \left[\frac{i}{n+1},\frac{i+1}{n+1}\right]\times\left[\frac{j}{n+1},\frac{j+1}{n+1}\right], \qquad j=1,\ldots,i, \quad i=1,\ldots,n, \quad n\geq 1\,. \tag{4.72}$$

The sequence (4.71) is bounded in $L_\infty([0,1]\times[0,1])$. The space $L_\infty([0,1]\times[0,1])$ is the dual space of $L_1([0,1]\times[0,1])$ hence, it has a corresponding weak*–topology. Since the space $L_1([0,1]\times[0,1])$ is separable, it follows now by Alaoglu's theorem, see, e.g. [K, pg. 140] or [RR, Theorem 6.62], that (4.71) converges in the weak*–topology to a function $\tilde{k}(x,y)\in L_\infty([0,1]\times[0,1])$. The uniform in $p\in\mathbb{N}$ weak convergence in each $L_p([0,1]\times[0,1])\supset L_\infty([0,1]\times[0,1])$, immediately follows.

Remark 3. Alternatively, using the Eberlein–Shmulyan theorem see, e.g., [Y, pg. 141], one finds that (4.71) has a weakly convergent subsequence in each $L_p([0,1]\times[0,1])$ space for $1<p<\infty$ with L^p–norms bounded uniformly in p. Using diagonal process we choose a subsequence $m(n)\in\mathbb{N}$ such that $\left\{k_{m(n)}(x,y)\right\}_{n\geq 1}$ converges weakly to the same function $\tilde{k}(x,y)$ in each of the spaces $L_p([0,1]\times[0,1])$, $p\in\mathbb{N}$. The function $\tilde{k}(x,y)$ along with $\left\{k_{m(n)}(x,y)\right\}_{n\geq 1}$ is uniformly bounded in all these L_p–spaces with the same bound for all $p\in\mathbb{N}$.

Remark 4. In the case of constant λ we have equality in (4.52). The right hand side is strictly monotone increasing in i, which results in $\tilde{k} \in C([0,1]; L_\infty(0,1))$.

Lemma 3. *The map* $\tilde{k} : [0,1] \to L_\infty(0,1)$ *given by* $x \longmapsto \tilde{k}(x,\cdot)$ *is weakly continuous.*

Proof. From the uniform boundedness in i of (4.52) we obtain that

$$\sum_{j=1}^{[nx]} k_{[nx],j} u_j = \sum_{j=1}^{[nx]} \left((n+1)\, k_{[nx],j}\right) u_j \frac{1}{n+1} \xrightarrow{n\to\infty} \int_0^x \tilde{k}(x,\xi)\, u(\xi)\, d\xi \qquad \forall u \in L_1(0,1), \quad \forall x \in [0,1]. \quad (4.73)$$

Here $[nx]$ denotes the largest integer not larger than nx and the convergence is uniform in x, meaning that for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \int_0^x \tilde{k}(x,\xi)\, u(\xi)\, d\xi - \sum_{j=1}^{[nx]} k_{[nx],j} u_j \right| < \varepsilon \qquad \forall x \in [0,1], \quad \forall n > N.$$

For an arbitrary $x \in [0,1]$ we now fix an $n > N(\varepsilon/2)$ and choose a $\delta > 0$ such that $[nx] = [n(x+\delta)]$. We obtain

$$\begin{aligned}
&\left| \int_0^1 \tilde{k}(x,\xi)\, u(\xi)\, d\xi - \int_0^1 \tilde{k}(x+\delta,\xi)\, u(\xi)\, d\xi \right| \\
&\leq \left| \int_0^x \tilde{k}(x,\xi)\, u(\xi)\, d\xi - \sum_{j=1}^{[nx]} k_{[nx],j} u_j \right| + \left| \sum_{j=1}^{[nx]} k_{[nx],j} u_j - \sum_{j=1}^{[n(x+\delta)]} k_{[n(x+\delta)],j} u_j \right| \\
&+ \left| \sum_{j=1}^{[n(x+\delta)]} k_{[n(x+\delta)],j} u_j - \int_0^{x+\delta} \tilde{k}(x+\delta,\xi)\, u(\xi)\, d\xi \right| \\
&< \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon
\end{aligned} \quad (4.74)$$

which proves weak continuity of $\tilde{k}$ from the right. For an arbitrary $x \in [0,1]$ we now fix an $n > N(\varepsilon/2)$ such that $[nx] \neq nx$ and choose a $\delta < 0$ such that $[nx] = [n(x+\delta)]$. Inequality (4.74) holds again, proving weak continuity from left. With this we obtain the statement of the lemma, i.e.

$$\tilde{k} \in C_w([0,1]; L_\infty(0,1)). \quad (4.75)$$

□

The following lemma shows how norms change under the above transformation.

Lemma 4 (Balogh and Krstić [BK1]). *Suppose that two functions* $w(x) \in L_\infty(0,1)$ *and* $u(x) \in L_\infty(0,1)$ *satisfy the relationship*

$$w(x) = u(x) - \int_0^x \tilde{k}(x,\xi)\, u(\xi)\, d\xi \qquad \forall x \in [0,1], \quad (4.76)$$

where

$$\tilde{k} \in C_w([0,1]; L_\infty(0,1)). \quad (4.77)$$

Then there exist positive constants m *and* M*, whose sizes depend only on* $\tilde{k}$*, such that*

$$m\, \|w\|_\infty \leq \|u\|_\infty \leq M\, \|w\|_\infty$$

and

$$m\, \|w\|_2 \leq \|u\|_2 \leq M\, \|w\|_2 .$$

Proof of Theorem 1. We now complete the proof of Theorem 1 by combining the results of Lemmas 1–4. In Lemma 1 we derived a coordinate transformation that transforms the finite dimensional system (2.6)–(2.8) into the finite dimensional system (2.12)–(2.14). As a result of the uniform boundedness of the transformation (shown in Lemma 2) we obtained the coordinate transformation (4.76) that transforms the system (2.5), (1.2) into the asymptotically stable system (2.15)–(2.17). Due to the weak continuity proven in Lemma 3 the infinite dimensional coordinate transformation results in the specific boundary condition

$$u(t,1) = \alpha(u) = \int_0^1 k(\xi)\, u(t,\xi)\, d\xi, \tag{4.78}$$

where

$$k(\xi) = \tilde{k}(1,\xi), \qquad \xi \in [0,1] \tag{4.79}$$

with $k \in L_\infty(0,1)$.

Figures 1 and 2 suggest the existence of smooth upper and lower envelopes to the strongly oscillating approximating kernel functions. This, in turn, could mean that the kernel function $k(x)$ coincides with the average of these smooth functions and hence it is smooth itself, at least in this simple case of constant coefficients. However, a kernel function in $L_\infty(0,1)$ is sufficient for us both in theory and in practice.

The convergence in Sobolev spaces $W_2^{2,1}$ (see, e.g. [A1]) of the finite difference approximations obtained from (2.6)–(2.8) and (2.12)–(2.14) to the solutions of (1.1)–(1.4) and (2.15)–(2.17) respectively is obtained using interpolation techniques (see, e.g. [BJ].) Using Green's function and fixed point method as it was done in [L1], we see that solutions to (1.1)–(1.4),(4.78) are, in fact, classical solutions.

Introducing a variable change

$$s(t,x) = w(t,x) e^{\frac{B}{2\varepsilon}x + \left(c + \frac{B^2}{4\varepsilon}\right)t} \tag{4.80}$$

we transform the w system (2.15)–(2.17) into a heat equation

$$s_t(t,x) = \varepsilon\, s_{xx}(t,x) \tag{4.81}$$

with homogeneous Dirichlet boundary conditions. The well known (see, e.g. [C1]) stability properties of the s system (4.81) along with Lemma 4 proves the stability statements of Theorem 1. □

4.2 Case 2: Neumann boundary condition at $x = 0$

In this section we prove Theorem 2. The proof is completely analogous to the proof of Theorem 1 and we only outline the differences.

Lemma 5. *The elements of the sequence $\{k_{i,j}\}$ defined in (2.43)–(2.46) satisfy*

$$\begin{aligned} |k_{i,i-j}| \;<\; & \binom{i}{j+1} L_n{}^{j+1} + (i-j) \sum_{l=1}^{[j/2]} \frac{1}{l} \binom{j-l}{l-1} \binom{i-l}{j-2l} L_n{}^{j-2l+1} M_n{}^{l} \\ & + 2 \sum_{l=0}^{[(j-1)/2]} \sum_{k=0}^{j-2l-1} \binom{l+k}{l} \binom{i-l-1}{k} M_n{}^{j-l-k} L_n{}^{k+1} \end{aligned} \tag{4.82}$$

where $\lambda = \max_{x\in[0,1]} |\lambda(x)|$.

Proof. The proof goes along the same lines as the proof of the Lemma 1. We first obtain estimates for the initial values of k's as

$$
\begin{aligned}
|k_{1,1}| &\le \frac{h^2}{\varepsilon+Bh}(c+\lambda) = L_n\,, && (4.83)\\
|k_{2,1}| &\le \left(\frac{h^2}{\varepsilon+Bh}(c+\lambda)+\frac{\varepsilon}{\varepsilon+Bh}\right)\frac{h^2}{\varepsilon+Bh}(c+\lambda) = {L_n}^2+M_nL_n\\
&\le {L_n}^2+2M_nL_n\,, && (4.84)\\
|k_{2,2}| &\le 2\frac{h^2}{\varepsilon+Bh}(c+\lambda) = 2L_n\,, && (4.85)\\
|k_{3,1}| &\le \left(\frac{h^2}{\varepsilon+Bh}(c+\lambda)+\frac{\varepsilon}{\varepsilon+Bh}\right)^2\frac{h^2}{\varepsilon+Bh}(c+\lambda)+\frac{\varepsilon}{(\varepsilon+Bh)}\frac{h^2}{(\varepsilon+Bh)}(c+\lambda)\\
&= (L_n+M_n)^2L_n+M_nL_n = L_n^3+2M_nL_n^2+M_n^2L_n+M_nL_n\\
&\le L_n^3+4M_nL_n^2+2M_n^2L_n+M_nL_n\,, && (4.86)\\
|k_{3,2}| &\le 3\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda)^2+\frac{\varepsilon}{\varepsilon+Bh}\frac{h^2}{\varepsilon+Bh}(c+\lambda) = 3{L_n}^2+M_nL_n\\
&\le 3{L_n}^2+2M_nL_n\,, && (4.87)\\
|k_{3,3}| &\le 3\frac{h^2}{\varepsilon+Bh}(c+\lambda) = 3L_n\,, && (4.88)
\end{aligned}
$$

and for $k_{i,i}$ and $k_{i,i-1}$ as

$$
\begin{aligned}
|k_{i,i}| &\le i\frac{h^2}{\varepsilon+Bh}(c+\lambda) = iL_n\,, && (4.89)\\
|k_{i,i-1}| &\le \frac{i(i-1)}{2}\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda)^2+\frac{\varepsilon}{\varepsilon+Bh}\frac{h^2}{\varepsilon+Bh}(c+\lambda) = \frac{i(i-1)}{2}{L_n}^2+M_nL_n\\
&\le \frac{i(i-1)}{2}{L_n}^2+2M_nL_n\,. && (4.90)
\end{aligned}
$$

Finally we obtain inequality (4.82) of Lemma 5 using the general identity for $k_{i,j}$ and mathematical induction. □

The only thing left now is to prove the uniform boundedness of $(n+1)\,k_{n,n-qn}$, with the bound on $k_{n,n-qn}$ given by

$$
\begin{aligned}
|k_{n,n-j}| &= |k_{n,n-qn}|\\
&\le \binom{n}{qn+1}{L_n}^{qn+1}+(n-qn)\sum_{l=1}^{[qn/2]}\frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}{L_n}^{qn-2l+1}{M_n}^{l}\\
&\quad +2\sum_{l=0}^{[(qn-1)/2]}\sum_{k=0}^{qn-2l-1}\binom{l+k}{l}\binom{n-l-1}{k}{M_n}^{qn-l-k}{L_n}^{k+1}\,. && (4.91)
\end{aligned}
$$

Lemma 6. *The sequence* $\left\{(n+1)\,k_{n,j}\right\}_{j=1,\ldots,n,\,n\ge1}$ *remains bounded uniformly in* n *and* j *as* $n\to\infty$.

Proof. We can write, according to (4.91),

$$
\begin{aligned}
(n+1)\,|k_{n,n-qn}| \;\le\; & (n+1)\binom{n}{qn+1}\left(\frac{E}{(n+1)^2}\right)^{qn+1} \\
& +(n+1)\,(n-qn)\sum_{l=1}^{[qn/2]}\frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1} {M_n}^{l} \\
& +2\,(n+1)\sum_{l=0}^{[(qn-1)/2]}\;\sum_{k=0}^{qn-2l-1}\binom{l+k}{l}\binom{n-l-1}{k} \\
& \times\left(\frac{E}{(n+1)^2}\right)^{k+1} {M_n}^{qn-l-k}\,. \qquad (4.92)
\end{aligned}
$$

Since the first two terms are identical to terms appearing in the expression (4.67), we only have to estimate the third term from (4.92). Using the simple inequalities

$$
\binom{l+k}{l}=\binom{l+k}{k}\le(l+k)^k\;, \qquad (4.93)
$$

$$
\binom{n-l-1}{k}\le\binom{n}{k}\;, \qquad (4.94)
$$

and

$$
l+k\le l+k_{max}=l+qn-2l-1=qn-l-1\le qn<n+1\;, \qquad (4.95)
$$

we obtain

$$
\begin{aligned}
& (n+1)\sum_{l=0}^{[(qn-1)/2]}\;\sum_{k=0}^{qn-2l-1}\binom{l+k}{l}\binom{n-l-1}{k}\left(\frac{E}{(n+1)^2}\right)^{k+1}{M_n}^{qn-l-k} \\
& \le\frac{E}{n+1}\sum_{l=0}^{[(qn-1)/2]}\;\sum_{k=0}^{qn-2l-1}\left(\frac{l+k}{n+1}\right)^k\binom{n}{k}\left(\frac{E}{n+1}\right)^k\left(1+\frac{R}{n+1}\right)^{qn-l-k} \\
& \le\frac{E}{n+1}\left(1+\frac{R}{n}\right)^n\sum_{l=0}^{[(qn-1)/2]}\;\sum_{s=0}^{qn}\binom{n}{s}\left(\frac{E}{n}\right)^s 1^{n-s} \\
& \le\frac{E}{n+1}\,e^R\left(1+\frac{E}{n}\right)^n\sum_{l=0}^{[(qn-1)/2]}1 \\
& \le E e^{R+E}\,.
\end{aligned}
$$

This proves the lemma. □

5 Extension of the Result to the Neumann Type of Actuation

In previous sections we derived control laws of the Dirichlet type to stabilize the system. We show here briefly how to extend them to the Neumann case.

Dirichlet control $u(t,1)$ was obtained by setting $w(t,1)=0$ in the transformation

$$
w(t,x)=u(t,x)-\int_0^x\tilde{k}\,(x,\xi)\,u\,(\xi)\;d\xi,\quad x\in[0,1]\;. \qquad (5.96)
$$

If one uses $u_x(t,1)$ for feedback then the boundary condition of the target system at $x = 1$ will be

$$w_x(t,1) = C_1 w(t,1)\,, \tag{5.97}$$

which can be shown to be exponentially stabilizing for both $w(t,0) = 0$ and $w_x(t,0) = 0$ for sufficiently large $c > 0$. We obtain the expression for the Neumann actuation in the original u coordinates by implementing the Neumann boundary condition (5.97) as

$$u_x(t,1) = C_1 u(t,1) + \tilde{k}(1,1)\,u(t,1) + \int_0^1 \tilde{k}_x(1,\xi)\,u(\xi)\,d\xi - C_1 \int_0^1 \tilde{k}(1,\xi)\,u(\xi)\,d\xi, \quad x \in [0,1]\,, \tag{5.98}$$

where $\tilde{k}_x$ denotes a partial derivative with respect to the first variable.

In terms of the discretized original and target systems, and discrete coordinate transformation $\alpha_i = \sum_{j=1}^{i} k_{i,j} u_j$, $i = 1,\ldots,n$, the discretized equivalent ${u_x}^{dis}(t,1)$ of $u_x(t,1)$ is

$$\begin{aligned} {u_x}^{dis}(t,1) &= C_1 u_n + \frac{\alpha_n - \alpha_{n-1}}{h} - C_1 \alpha_n \\ &= C_1 u_n + \frac{k_{n,n}}{h} u_n + \sum_{j=1}^{n-1} \frac{k_{n,j} - k_{n-1,j}}{h} u_j - C_1 \sum_{j=1}^{n} k_{n,j} u_j\,. \end{aligned} \tag{5.99}$$

Comparing (5.99) and (5.98) term by term, it is evident that uniform boundedness of $\frac{k_{i,j}}{h}$ will guarantee that

$${u_x}^{dis}(t,1) \xrightarrow{n\to\infty} u_x(t,1) \tag{5.100}$$

for all $t > 0$.

6 Extension to the case with non–zero integral term on the RHS of the system equation

In previous sections we showed how to stabilize a less general case of the system (1.1) with no integral term on the RHS of the system equation. In this section we present the extension to

$$u_t(t,x) = \varepsilon u_{xx}(t,x) + B u_x(t,x) + \lambda(x)\,u(t,x) + \int_0^x f(x,\xi) u(t,\xi) d\xi\,, \qquad x \in (0,1), \quad t > 0 \tag{6.101}$$

with a homogeneous Dirichlet boundary condition at $x = 0$,

$$u(t,0) = 0\,, \qquad t > 0\,, \tag{6.102}$$

and Dirichlet boundary condition

$$u(t,1) = \alpha(u(t))\,, \qquad t > 0\,, \tag{6.103}$$

used for actuation at the other end. One immediately notices the overlap between the extension presented in this section and the results from Section 2.1 that are special case. A natural question to ask would be why the problem was not given in its most general form in Section 2.1. The reason is that as of now we have not succeeded in extending the results from Section 2.2 to the most general case, and it is easy to see why. By comparing Dirichlet and Neumann control results for the case without the integral term one can see that the expression for the kernel in the Neumann case (2.47) is much more complex than the Dirichlet one given by expression (2.31). The level of complexity increases even more when integral term is present, as it will be seen in this section, which prevented us from obtaining a closed form expression in the Neumann case with integral term. The only reason for presenting the less general Dirichlet result first was to draw the parallel between the cases for $u(t,0)=0$ and $u_x(t,0)=0$ and present only the differences.

As in Section 2.1 we choose discretization of (2.15)–(2.17) for our target system, obtain the recursive expression for the discretized backstepping–style coordinate transformation as

$$\begin{aligned}\alpha_i = \ & (\varepsilon+Bh)^{-1}\Bigg\{\left(2\varepsilon+Bh+ch^2\right)\alpha_{i-1}-\varepsilon\alpha_{i-2}-(\lambda_i+c)\,h^2u_i \\ & -h^3\sum_{k=1}^{i-1}f_{i,k}u_k+\frac{\partial\alpha_{i-1}}{\partial u_1}\left((\varepsilon+Bh)\,u_2-\left(2\varepsilon+Bh-\lambda_1h^2\right)u_1\right) \\ & +\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial u_j}\left((\varepsilon+Bh)u_{j+1}-\left(2\varepsilon+Bh-\lambda_jh^2\right)u_j+\varepsilon u_{j-1}+h^3\sum_{k=1}^{j-1}f_{j,k}u_k\right)\Bigg\},\end{aligned} \tag{6.104}$$

and then, assuming the linear form for α_i's ($\alpha_i=\sum_{j=1}^{i}k_{i,j}u_j$), obtain the general recursive relationship for the kernel as

$$\begin{aligned} k_{i,1} &= \frac{h^2}{\varepsilon+Bh}(c+\lambda_1)\,k_{i-1,1}+\frac{\varepsilon}{\varepsilon+Bh}(k_{i-1,2}-k_{i-2,1}) \\ &\quad -\frac{h^3}{\varepsilon+Bh}f_{i,1}+\frac{h^3}{\varepsilon+Bh}\sum_{l=2}^{i-1}f_{l,1}k_{i-1,l}\,, \end{aligned} \tag{6.105}$$

$$\begin{aligned} k_{i,j} &= \frac{h^2}{\varepsilon+Bh}(c+\lambda_j)\,k_{i-1,j}+k_{i-1,j-1}+\frac{\varepsilon}{\varepsilon+Bh}(k_{i-1,j+1}-k_{i-2,j}) \\ &\quad -\frac{h^3}{\varepsilon+Bh}f_{i,j}+\frac{h^3}{\varepsilon+Bh}\sum_{l=j+1}^{i-1}f_{l,j}k_{i-1,l}\,, \quad j=2,\dots,i-2, \end{aligned} \tag{6.106}$$

$$k_{i,i-1} = \frac{h^2}{\varepsilon+Bh}(c+\lambda_{i-1})\,k_{i-1,i-1}+k_{i-1,i-2}-\frac{h^3}{\varepsilon+Bh}f_{i,i-1}\,, \tag{6.107}$$

$$k_{i,i} = k_{i-1,i-1}-\frac{h^2}{\varepsilon+Bh}(c+\lambda_i)\,, \tag{6.108}$$

for $i=4,\dots,n$ with initial conditions

$$k_{1,1} = -\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)\,, \tag{6.109}$$

$$k_{2,1} = -\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda_1)^2-\frac{h^3}{\varepsilon+Bh}f_{2,1}\,, \tag{6.110}$$

$$k_{2,2} = -\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\right), \tag{6.111}$$

$$\begin{aligned} k_{3,1} &= -\frac{h^6}{(\varepsilon+Bh)^3}(c+\lambda_1)^3-\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)\frac{h^3}{\varepsilon+Bh}f_{2,1}-\frac{\varepsilon}{\varepsilon+Bh}\frac{h^2}{\varepsilon+Bh}(c+\lambda_2) \\ &\quad -\frac{h^3}{\varepsilon+Bh}f_{3,1}-\frac{h^3}{\varepsilon+Bh}\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\right)f_{2,1}\,, \end{aligned} \tag{6.112}$$

$$\begin{aligned} k_{3,2} &= -\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)\right) \\ &\quad -\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda_1)^2-\frac{h^3}{\varepsilon+Bh}f_{2,1}-\frac{h^3}{\varepsilon+Bh}f_{3,2}\,, \end{aligned} \tag{6.113}$$

$$k_{3,3} = -\left(\frac{h^2}{\varepsilon+Bh}(c+\lambda_1)+\frac{h^2}{\varepsilon+Bh}(c+\lambda_2)+\frac{h^2}{\varepsilon+Bh}(c+\lambda_3)\right). \tag{6.114}$$

For the simple case when $\lambda(x)\equiv\lambda$ = constant and $f(x,y)\equiv f$ = constant, equations (6.105)–(6.114) can be solved explicitly to obtain

$$\begin{aligned} k_{i,i-j} &= -\binom{i}{j+1}L_n{}^{j+1}-(i-j)\sum_{l=1}^{[j/2]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}L_n{}^{j-2l+1}M_n{}^l \\ &\quad -(i-j)\sum_{l=1}^{[(j+1)/2]}\frac{1}{l}P_n{}^l\sum_{m=0}^{[(j+1)/2]-l}\binom{l+m-1}{l-1}M_n{}^m \\ &\quad \times\sum_{k=0}^{j+1-2l-2m}\binom{j-l-2m-k}{l-1}\binom{k+l+m-1}{k}\binom{i-m}{k+l-1}L_n{}^k \end{aligned} \tag{6.115}$$

for $i=1,\ldots,n,\;\; j=0,\ldots,i-1$, where

$$P_n=\frac{h^3}{\varepsilon+Bh}f\,. \tag{6.116}$$

The precise formulation of the main result in this section is summarized in the following theorem.

Theorem 3. *For any $\lambda(x)\in L_\infty(0,1)$, $f(x,y)\in L_\infty([0,1]\times[0,1])$, and $\varepsilon, c>0$ there exists a function $k\in L_\infty(0,1)$ such that for any $u^0\in L_\infty(0,1)$ the unique classical solution $u(t,x)\in C^1\left((0,\infty);C^2(0,1)\right)$ of system (6.101)–(6.103), (2.34) is exponentially stable in the $L_2(0,1)$ and maximum norms with decay rate c. The precise statements of stability properties are the following: There exists positive constant M*[¶] *such that for all $t>0$*

$$\|u(t)\|_2\le M\left\|u^0\right\|_2 e^{-ct} \tag{6.117}$$

and

$$\max_{x\in[0,1]}|u(t,x)|\le M\sup_{x\in[0,1]}\left|u^0(x)\right|e^{-ct}. \tag{6.118}$$

The proof of Theorem 3 is completely analogous to the proof of Theorem 1 and we only outline the differences.

Lemma 7. *The elements of the sequence $\{k_{i,j}\}$ defined in (6.105)–(6.114) satisfy*

$$\begin{aligned} |k_{i,i-j}| &\le \binom{i}{j+1}L_n{}^{j+1}+(i-j)\sum_{l=1}^{[j/2]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}L_n{}^{j-2l+1}M_n{}^l \\ &\quad +(i-j)\sum_{l=1}^{[(j+1)/2]}\frac{1}{l}P_n{}^l\sum_{m=0}^{[(j+1)/2]-l}\binom{l+m-1}{l-1}M_n{}^m \\ &\quad \times\sum_{k=0}^{j+1-2l-2m}\binom{j-l-2m-k}{l-1}\binom{k+l+m-1}{k}\binom{i-m}{k+l-1}L_n{}^k \end{aligned} \tag{6.119}$$

where $\lambda=\max_{x\in[0,1]}|\lambda(x)|$ and $f=\sup_{(x,y)\in[0,1]\times[0,1]}|f(x,y)|$.

[¶]M grows with c, λ and $1/\varepsilon$.

Remark 5. There is equality in (6.119) when $\lambda(x) \equiv \lambda$ = constant > 0 and $f(x,y) \equiv f$ = constant > 0.

Proof. The right hand side of equations (6.109)–(6.114) can be estimated to obtain estimates for the initial values of k's

$$
\begin{aligned}
|k_{1,1}| &\leq \frac{h^2}{\varepsilon+Bh}(c+\lambda) = L_n\,, && (6.120)\\
|k_{2,1}| &\leq \frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda)^2 + \frac{h^3}{\varepsilon+Bh} f = {L_n}^2 + P_n\,, && (6.121)\\
|k_{2,2}| &\leq 2\frac{h^2}{\varepsilon+Bh}(c+\lambda) = 2L_n\,, && (6.122)\\
|k_{3,1}| &\leq \frac{h^6}{(\varepsilon+Bh)^3}(c+\lambda)^3 + 3\frac{h^2}{\varepsilon+Bh}(c+\lambda)\frac{h^3}{\varepsilon+Bh} f + \frac{\varepsilon}{\varepsilon+Bh}\frac{h^2}{\varepsilon+Bh}(c+\lambda)\\
&\quad + \frac{h^3}{\varepsilon+Bh} f = {L_n}^3 + 3L_nP_n + M_nL_n + P_n\,, && (6.123)\\
|k_{3,2}| &\leq 3\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda)^2 + 2\frac{h^3}{\varepsilon+Bh} f = 3{L_n}^2 + 2P_n\,, && (6.124)\\
|k_{3,3}| &\leq 3\frac{h^2}{\varepsilon+Bh}(c+\lambda) = 3L_n\,. && (6.125)
\end{aligned}
$$

We then go from $j = i$ backwards and obtain

$$
\begin{aligned}
|k_{i,i}| &\leq i\frac{h^2}{\varepsilon+Bh}(c+\lambda) = iL_n\,, && (6.126)\\
|k_{i,i-1}| &\leq \frac{i(i-1)}{2}\frac{h^4}{(\varepsilon+Bh)^2}(c+\lambda)^2 + (i-1)\frac{h^3}{\varepsilon+Bh} f\\
&= \frac{i(i-1)}{2}{L_n}^2 + (i-1)P_n\,, && (6.127)
\end{aligned}
$$

from (6.107) and (6.108) respectively. Finally we obtain inequality (6.119) of Lemma 7 using the general identity (6.106) and mathematical induction. □

To prove the uniform boundedness of $(n+1)k_{n,n-qn}$ we start by finding a bound on $k_{n,n-qn}$ as

$$
\begin{aligned}
|k_{n,n-j}| &= |k_{n,n-qn}|\\
&\leq \binom{n}{qn+1}{L_n}^{qn+1} + (n-qn)\sum_{l=1}^{[qn/2]}\frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}{L_n}^{qn-2l+1}{M_n}^{l}\\
&\quad + (n-qn)\sum_{l=1}^{[(qn+1)/2]}\frac{1}{l}{P_n}^{l}\sum_{m=0}^{[(qn+1)/2]-l}\binom{l+m-1}{l-1}{M_n}^{m}\\
&\quad \times \sum_{k=0}^{qn+1-2l-2m}\binom{qn-l-2m-k}{l-1}\binom{k+l+m-1}{k}\binom{n-m}{k+l-1}{L_n}^{k}\,,
\end{aligned}
\tag{6.128}
$$

where

$$
P_n = \frac{h^3}{\varepsilon+Bh} f \leq \frac{|f|h^3}{\frac{\varepsilon}{2}} = \frac{H}{(n+1)^3}\,, \tag{6.129}
$$

H being defined as

$$H = \frac{2|f|}{\varepsilon}\,. \tag{6.130}$$

The uniform boundedness of the kernel is given by the following lemma.

Lemma 8. *The sequence* $\left\{(n+1)\,k_{n,j}\right\}_{j=1,\ldots,n,\; n\geq 1}$ *remains bounded uniformly in n and j as* $n\to\infty$.

Proof. We can write, according to (6.128),

$$\begin{aligned}
& (n+1)\,|k_{n,n-qn}| \\
\leq\ & (n+1)\binom{n}{qn+1}\left(\frac{E}{(n+1)^2}\right)^{qn+1} \\
& +(n+1)\,(n-qn)\sum_{l=1}^{[qn/2]}\frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1}{M_n}^l \\
& +(n+1)\,(n-qn)\sum_{l=1}^{[(qn+1)/2]}\frac{1}{l}{P_n}^l\sum_{m=0}^{[(qn+1)/2]-l}\binom{l+m-1}{l-1}{M_n}^m \\
& \times\sum_{k=0}^{qn+1-2l-2m}\binom{qn-l-2m-k}{l-1}\binom{k+l+m-1}{k}\binom{n-m}{k+l-1}\left(\frac{E}{(n+1)^2}\right)^k
\end{aligned} \tag{6.131}$$

Since the first two terms are identical to terms appearing in the expression (4.67), we only have to estimate the third term in (6.131). We start with simple inequalities

$$\binom{n-m}{k+l-1} \leq \binom{n}{k}\,, \tag{6.132}$$

$$\frac{\binom{qn-l-2m-k}{l-1}}{(n+1)^{l-1}} \leq 1\,, \tag{6.133}$$

$$\frac{\binom{k+l+m-1}{k}}{(n+1)^k} \leq 1\,, \tag{6.134}$$

and obtain

$$\begin{aligned}
& (n{+}1)\,(n{-}qn)\sum_{l=1}^{[(qn+1)/2]}\frac{1}{l}{P_n}^l\sum_{m=0}^{[(qn+1)/2]-l}\binom{l+m-1}{l-1}{M_n}^m \\
& \times\sum_{k=0}^{qn+1-2l-2m}\binom{qn-l-2m-k}{l-1}\binom{k+l+m-1}{k}\binom{n-m}{k+l-1}\left(\frac{E}{(n+1)^2}\right)^k \\
\leq\ & (n+1)^2\sum_{l=1}^{[(qn+1)/2]}\frac{1}{l}\frac{H^l}{(n+1)^{2l+1}}\sum_{m=0}^{[(qn+1)/2]-l}\left(1+\frac{R}{n}\right)^m\binom{l+m-1}{l-1} \\
& \times\sum_{k=0}^{qn+1-2l-2m}\frac{\binom{qn-l-2m-k}{l-1}}{(n+1)^{l-1}}\,\frac{\binom{k+l+m-1}{k}}{(n+1)^k}\binom{n-m}{k+l-1}\left(\frac{E}{n+1}\right)^k
\end{aligned}$$

$$\begin{aligned}
&\leq H \sum_{l=1}^{[(qn+1)/2]} \frac{H^{l-1}}{(n+1)^{2l-1}} \sum_{m=0}^{[(qn+1)/2]-l} \left(1+\frac{R}{n}\right)^m \binom{l+m-1}{l-1} \\
&\quad \sum_{k=0}^{qn+1-2l-2m} \binom{n}{k} \left(\frac{E}{n+1}\right)^k 1^{n-k} \\
&\leq H\left(1+\frac{R}{n}\right)^n \sum_{l=1}^{[(qn+1)/2]} \frac{H^{l-1}}{(n+1)^{2l-1}} \binom{\left[\frac{qn+1}{2}\right]-1}{l-1} \sum_{m=0}^{[(qn+1)/2]-l} 1 \sum_{k=0}^{n} \binom{n}{k} \left(\frac{E}{n}\right)^k 1^{n-k} \\
&\leq H\left(1+\frac{R}{n}\right)^n \left(1+\frac{E}{n}\right)^n \sum_{l=1}^{[(qn+1)/2]} \binom{\left[\frac{qn+1}{2}\right]-1}{l-1} \left(\frac{H}{n}\right)^{l-1} \frac{\left[\frac{qn+1}{2}\right]-l+1}{(n+1)^l} \\
&\leq H\left(1+\frac{R}{n}\right)^n \left(1+\frac{E}{n}\right)^n \sum_{l=1}^{n} \binom{n-1}{l-1} \left(\frac{H}{n}\right)^{l-1} \frac{n}{(n+1)^l} \\
&\leq H\left(1+\frac{R}{n}\right)^n \left(1+\frac{E}{n}\right)^n \sum_{l=0}^{n} \binom{n-1}{l} \left(\frac{H}{n}\right)^{l-1} 1^{n-l-1} \\
&\leq H\left(1+\frac{R}{n}\right)^n \left(1+\frac{E}{n}\right)^n \left(1+\frac{H}{n}\right)^n \leq He^{(R+E+H)} .
\end{aligned}$$

This proves the lemma. □

7 Simulation Study

In this section we present the simulation results for a linearization of an adiabatic chemical tubular reactor. For the case when Peclet numbers for heat and mass transfer are equal (Lewis number of unity) the two equations for the temperature and concentration can be reduced to one equation [HH1]

$$\theta_t = \frac{1}{Pe}\theta_{\xi\xi} - \theta_\xi + Da\,(b-\theta)\,e^{\frac{\theta}{1+\mu\theta}}\,, \qquad \xi \in (0,1), \quad t>0\,, \tag{7.135}$$

$$\theta_\xi\,(t,0) = Pe\,\theta\,(t,0)\,, \tag{7.136}$$

$$\theta_\xi\,(t,1) = 0\,, \tag{7.137}$$

where Pe stands for the Peclet number, Da for the Damköhler number, μ for the dimensionless activation energy, and b for the dimensionless adiabatic temperature rise. For a particular choice of system parameters ($Pe = 6$, $Da = 0.05$, $\mu = 0.05$, and $b = 10$) system (7.135)–(7.137) has three equilibria [HH2]. As shown in [HH2], the middle profile is unstable while the outer two profiles are stable. The equilibrium profiles for this case are shown in Figure 3. Linearizing the system around the unstable equilibrium profile $\bar{\theta}(\xi)$ we obtain

$$\theta_t = \frac{1}{Pe}\theta_{\xi\xi} - \theta_\xi + Da\,G\left(\bar{\theta}(\xi)\right)\theta\,, \tag{7.138}$$

$$\theta_\xi\,(t,0) = Pe\theta\,(t,0)\,, \tag{7.139}$$

$$\theta_\xi\,(t,1) = 0\,, \tag{7.140}$$

where θ now stands for the deviation variable from the steady state $\bar{\theta}(\xi)$, and G is a spatially dependent coefficient defined as

$$G(\bar{\theta}) = \left[\frac{b-\bar{\theta}}{(1+\mu\bar{\theta})^2} - 1\right] e^{\frac{\bar{\theta}}{1+\mu\bar{\theta}}}\,. \tag{7.141}$$

Although not obvious from the equations (7.138)–(7.140), it is physically justifiable to apply feedback boundary control at 0-end only. In real application control would be implemented through small variations of the inlet temperature and the inlet reactant concentration (see [VA] and [HH1]). Since our control algorithm assumes actuation at 1-end we transform the original system (7.138)–(7.140) by introducing a variable change

$$u(t,x) = \theta(1-\xi)\,. \tag{7.142}$$

In the new set of variables the system (7.138)–(7.140) becomes

$$\begin{aligned} u_t(t,x) &= \frac{1}{Pe}u_{xx}(t,x) + u_x(t,x) + Da\, g(x)\, u(t,x)\,, & (7.143)\\ u_x(t,0) &= 0\,, & (7.144)\\ u_x(t,1) &= -Pe u(t,1) + \Delta u_x(t,1)\,, & (7.145) \end{aligned}$$

where $g(x)$ is defined as

$$g(x) = G(\bar{\theta}(1-\xi))\,, \tag{7.146}$$

and $\Delta u_x(t,1)$ stands for the control law to be designed. All simulations presented in this study were done using BTCS finite difference method for $n = 200$ and the time step equal to 0.001 s. Although we have tested the controller for several different combinations of initial distributions and target systems, we only present results for $c = 0.1$ and $u(0,x) = -\left(\frac{\omega}{Pe}\cos(\omega x) + \sin(\omega x)\right)$, $\omega = 1.48396$. This particular initial distribution has been constructed to exactly satisfy the imposed boundary conditions on both ends in the open loop case.

As expected, since the system (7.143)–(7.145) represents a linearization around the unstable steady state, the open loop system ($\Delta u_x(t,1)$=0) is unstable, and state grows exponentially as shown in Figure 4. We now apply approach outlined in Subsection 2.2 and obtain a coordinate transformation that transforms the discretization of (7.143)–(7.145) into discretization of the asymptotically stable system

$$\begin{aligned} w_t(t,x) &= \frac{1}{Pe}w_{xx}(t,x) + w_x(t,x) - cw(t,x)\,, & (7.147)\\ w_x(t,0) &= 0\,, & (7.148)\\ w_x(t,1) &= -Pe w(t,1)\,. & (7.149) \end{aligned}$$

The control is implemented as

$$\Delta u_x(t,1) = \frac{\alpha_n(u_1,\cdots,u_n) - \alpha_{n-1}(u_1,\cdots,u_{n-1})}{h} + Pe\alpha_n(u_1,\cdots,u_n), \tag{7.150}$$

where h stands for the discretization step in the controller design. The closed loop response of the system with a controller designed for $n = 200$ and $c = 0.1$ and the corresponding control effort $\Delta u_x(t,1)$ are shown in Figure 5.

From applications point of view it would of interest to see whether the system (7.143)–(7.145) could be stabilized with a reduced version of the control law (7.150). By a reduced order controller we assume a controller designed on a much coarser grid than the one used for simulating the response of the system. The expectation that the system might be rendered stable with a low order backstepping controller is based on our past experience in designing nonlinear low order backstepping controllers for the heat convection loop [BK4], stabilization of unstable burning in solid propellant rockets [BK5], and stabilization of chemical tubular reactors [BK2]. The idea of using controllers designed using only a small number of steps of backstepping to stabilize the system for a certain range of the open–loop instability is based on the fact that in most real life systems only a finite number of open–loop eigenvalues is unstable. The conjecture is then to apply a low order backstepping controller (controller that uses only a small number of state measurements) that is capable of detecting the occurrence of instability from a limited number of measurements, and stabilize the system. Indeed, simulation results show that we can successfully stabilize the unstable equilibrium using a kernel obtained with only two steps of backstepping (using only two state measurements $u(t,\frac{1}{3})$ and $u(t,\frac{2}{3})$) with the same $c = 0.1$. By controller designed using only two steps of backstepping we assume controller designed on a very coarse grid, namely on a grid with just three points. In this case the control is implemented by substituting α_1 and α_2 in expression (7.150) for $\Delta u_x(t,1)$, where α_1 and α_2 are obtained from expressions (2.25), (2.44), and (2.27) with $h = \frac{1}{3}$, $\varepsilon = \frac{1}{Pe}$, $B = 1$, $\lambda_1 = Da\, g(\frac{1}{3})$, $\lambda_2 = Da\, g(\frac{2}{3})$, $u_1 = u(t,\frac{1}{3})$, and $u_2 = u(t,\frac{2}{3})$. The closed loop response of the system with a reduced order controller and corresponding control effort $\Delta u_x(t,1)$ are shown in Figure 6.

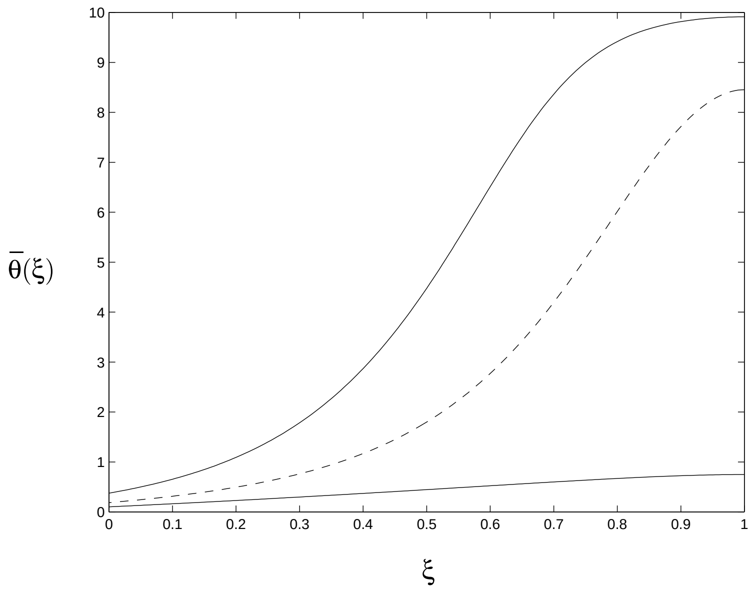

Figure 3: Steady state profiles for the adiabatic chemical tubular reactor with $Pe = 6$, $Da = 0.05$, $\mu = 0.05$ and $b = 10$.

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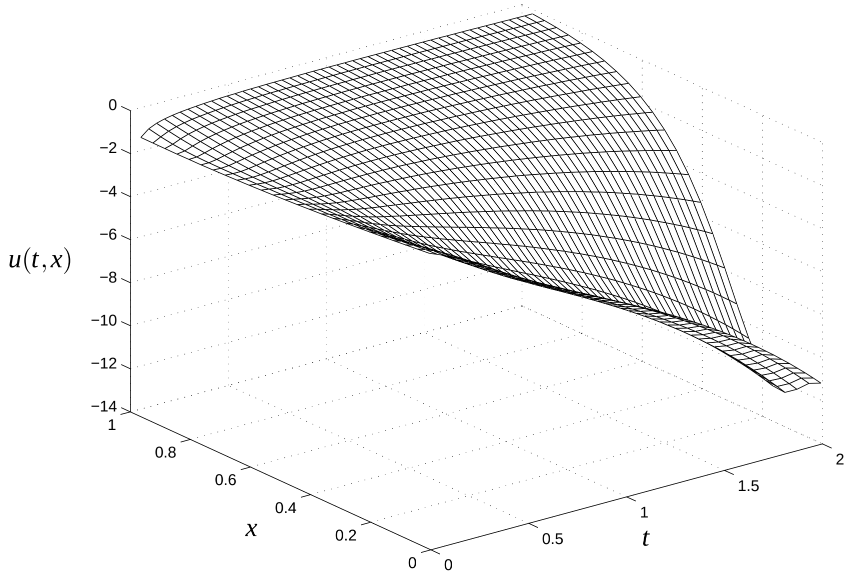

Figure 4: Open loop response of the system.

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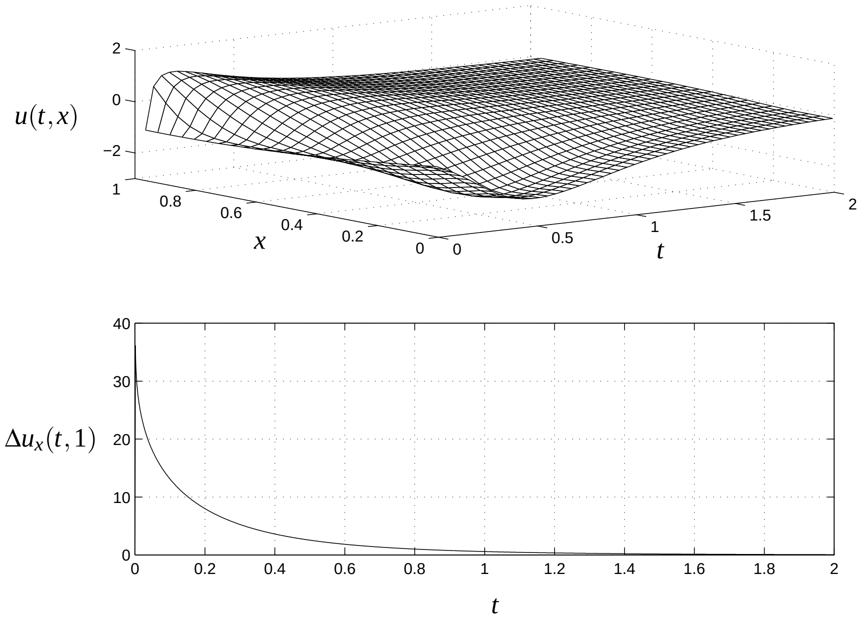

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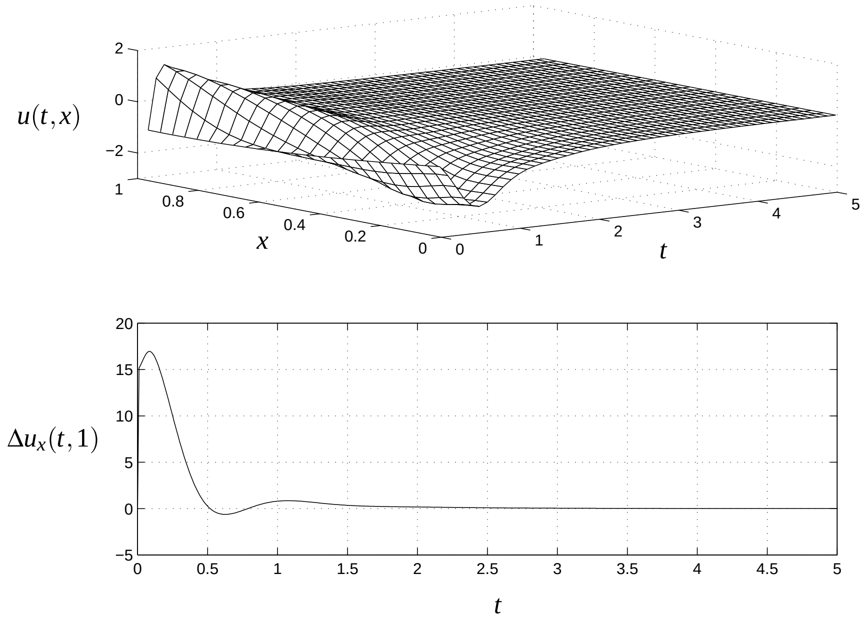

Figure 6: Closed loop response of the system with controller designed using only two steps of backstepping. (First row: $u(t,x)$; Second row: The control effort $\Delta u_x(t,1)$.)

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