Problem Three

Infinite Dimensional Backstepping for Nonlinear Parabolic PDEs

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3.1 INTRODUCTION

This note explores an approach to global stabilization of boundary controlled nonlinear PDEs by a technique inspired by finite dimensional backstepping/feedback linearization. Solution of the problem presented herein would be of enormous significance because these are the only truly constructive and systematic techniques in finite dimension.

We consider nonlinear parabolic PDE's of the form

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + f(u(x,t)) \tag{3.1}$$

for $x \in (0,1)$, $t > 0$, with boundary conditions

$$u(0,t) = 0, \tag{3.2}$$

$$u(1,t) = \alpha_1(u), \tag{3.3}$$

initial condition

$$u(x,0) = u_0(x), \qquad x \in [0,1],$$

and under the assumption

$$\varepsilon > 0, \quad f \in C^\infty(\mathbb{R}).^1 \tag{3.4}$$

The task is to derive a nonlinear (feedback) functional $\alpha_1 : C([0,1]) \to \mathbb{R}$ that stabilizes the trivial solution $u(x,t) \equiv 0$ in an appropriate way. An infinite dimensional version of backstepping was introduced in [2] that solves

[1] The smoothness requirement is explained after formula (3.18).

this problem for $f(u) = \lambda u$ with $\lambda > 0$ arbitrarily large. Superlinear nonlinearities can imply finite time blow–up for the uncontrolled case [6, 7, 9, 10]. However, numerical results in a series of papers by Boskovic and Krstic [3, 4, 5] show promise for the applicability of the infinite dimensional backstepping to nonlinear problems, at least for finite–grid discretizations. In this note we present the open problem of convergence of nonlinear backstepping schemes as the discretization grid becomes infinitely refined. Note that this problem is different from the question of controllability [1, 8].

3.2 BACKSTEPPING TRANSFORMATION

We look for a coordinate transformation of the form

$$w = u - \alpha(u), \tag{3.5}$$

where $\alpha : C([0,1]) \to C([0,1])$ is a nonlinear operator to be found, that transforms system (3.1)–(3.3) into the exponentially stable system

$$w_t(x,t) = \varepsilon w_{xx}(x,t), \qquad x \in (0,1), \quad t > 0, \tag{3.6}$$

with boundary conditions

$$w(0,t) = 0, \tag{3.7}$$

$$w(1,t) = 0. \tag{3.8}$$

Once transformation (3.5) is found, it is realized through the stabilizing boundary feedback control (3.3) with $\alpha_1(u) = \alpha(u)|_{x=1}$.

In order to find (3.5) in a constructive way we first discretize in space (3.1)–(3.3), then we develop a stabilizing coordinate transformation for the semi–discretized system. The main question of showing that the discretization converges to an infinite dimensional transformation is open in the case of functions $f(u)$ that are nonlinear.

We define $u_i^n = u(ih, t)$ for $i, j = 0, 1, \ldots, n+1$, $n = 1, 2, \ldots$ where $h = 1/(n+1)$, and the finite difference discretization of the rest of the functions is defined the same way. The discretized version of coordinate transformation (3.5) now has the form

$$\mathbf{w}^n = (\mathbf{I} - \alpha^n)(\mathbf{u}^n) \qquad n = 1, 2, \ldots \tag{3.9}$$

where α^n is an n–vector valued function of $\mathbf{u}^n$ and

$$\mathbf{w}^n = \left[w_0^n, w_1^n, \ldots, w_{n+1}^n\right]^T, \tag{3.10}$$

$$\mathbf{u}^n = \left[u_0^n, u_1^n, \ldots, u_{n+1}^n\right]^T. \tag{3.11}$$

The discretized form of system (3.1)–(3.3) is

$$u_0^n = 0, \tag{3.12}$$

$$\dot{u}_i^n = \varepsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + f(u_i^n), \qquad i = 1, \ldots, n, \tag{3.13}$$

$$u_{n+1}^n = \alpha_n^n(u_1^n, u_2^n, \ldots, u_n^n). \tag{3.14}$$

with the convention of $\alpha_0^n = 0$. The discretized form of system (3.6)–(3.8) is

$$w_0^n = 0\,, \tag{3.15}$$

$$\dot{w}_i^n = \varepsilon \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2}, \qquad i = 1, 2, \ldots, n\,, \tag{3.16}$$

$$w_{n+1}^n = 0\,. \tag{3.17}$$

Combining (3.16), (3.9) and (3.13), and solving for α_i^n we obtain the final form of the recursive formula for the transformation:

$$\alpha_i^n = -\frac{h^2}{\varepsilon} f\left(u_i^n\right) + 2\alpha_{i-1}^n - \alpha_{i-2}^n + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}^n}{\partial u_j} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n + \frac{h^2}{\varepsilon} f\left(u_j^n\right) \right) \tag{3.18}$$

for $i = 1, 2, \ldots, n$. This recursive formula contains the functions $f(u)$ (which is nonlinear in general,) and it involves differentiation. As a result, as $n \to \infty$, eventually infinite smoothness of the function f is required. A few values of α_i^n:

$$\alpha_0^n = 0, \tag{3.19}$$

$$\alpha_1^n = -\frac{h^2}{\varepsilon} f\left(u_1^n\right), \tag{3.20}$$

$$\alpha_2^n = -\frac{h^2}{\varepsilon} f\left(u_2^n\right) - 2\frac{h^2}{\varepsilon} f\left(u_1^n\right) - \frac{h^2}{\varepsilon} f'\left(u_1^n\right) \left(u_2^n - 2u_1^n + \frac{h^2}{\varepsilon} f\left(u_1^n\right) \right), \tag{3.21}$$

$$\begin{aligned} \alpha_3^n = &-\frac{h^2}{\varepsilon} f\left(u_3^n\right) - 2\frac{h^2}{\varepsilon} f\left(u_2^n\right) - 3\frac{h^2}{\varepsilon} f\left(u_1^n\right) - 2\frac{h^2}{\varepsilon} f'\left(u_1^n\right) \left(u_2^n - 2u_1^n + \frac{h^2}{\varepsilon} f\left(u_1^n\right) \right) \\ &+ \left(-\frac{h^2}{\varepsilon} f''\left(u_1^n\right) \left(u_2^n - 2u_1^n + \frac{h^2}{\varepsilon} f\left(u_1^n\right) \right) - \left(\frac{h^2}{\varepsilon} f'\left(u_1^n\right) \right)^2 \right) \left(u_2^n - 2u_1^n + \frac{h^2}{\varepsilon} f\left(u_1^n\right) \right) \\ &- \left(\frac{h^2}{\varepsilon} f'\left(u_2^n\right) + \frac{h^2}{\varepsilon} f'\left(u_1^n\right) \right) \left(u_3^n - 2u_2^n + \frac{h^2}{\varepsilon} f\left(u_2^n\right) \right) \end{aligned} \tag{3.22}$$

3.3 OPEN PROBLEM

Using the above backstepping approach, the problem of finding the coordinate transformation (3.5) and the corresponding stabilizing boundary control (3.3) requires two steps.

1. Find assumptions on the nonlinear function f that ensures the convergence of the discretized coordinate transformation (3.18) to a (nonlinear) operator α in order to obtain the feedback boundary control law (3.5).

2. Establish the bounded invertibility of operator $I - \alpha$ (see equation (3.5)) in appropriate function spaces.

3.4 KNOWN LINEAR RESULT

For the linear case $f(u) = \lambda u$ we have the following result [2].

Theorem 3.1 *For any $\lambda \in \mathbb{R}$ and $\varepsilon, c > 0$ there exists a function $k_1 \in L_\infty(0,1)$ such that for any $u_0 \in L_\infty(0,1)$ the unique classical solution $u(x,t) \in C^1\big((0,\infty); C^2(0,1)\big)$ of system (3.1)–(3.3) with boundary feedback control*

$$\alpha_1(u) = \int_0^1 k_1(\xi)\, u(\xi,t)\, d\xi \tag{3.23}$$

is exponentially stable in the $L_2(0,1)$ and maximum norms with decay rate c. The precise statements of stability properties are the following: There exists a positive constant M[2] such that for all $t > 0$

$$\|u(t)\| \le M\, \|u_0\|\, e^{-ct} \tag{3.24}$$

and

$$\max_{x\in[0,1]} |u(t,x)| \le M \sup_{x\in[0,1]} |u_0(x)|\, e^{-ct}. \tag{3.25}$$

In this linear case the transformation is a bounded linear operator $\alpha : L_1 \to L_1$ in the form of $\alpha(u) = \int_0^x k(x,\xi)\, u(\xi)\, d\xi$ with integral kernel $k \in L_\infty([0,\infty] \times [0,\infty])$. The boundary control is $\alpha_1(u) = \int_0^1 k(1,\xi)\, u(\xi)\, d\xi$. The explicit form of α_i is

$$\alpha_i^n = \sum_{j=1}^{i} k_{i,j}^n u_j^n\,, \qquad i = 1, \ldots, n, \tag{3.26}$$

where

$$k_{i,i-j}^n = -\begin{pmatrix} i \\ j+1 \end{pmatrix} \left(\frac{(c+\lambda)}{\varepsilon\,(n+1)^2} \right)^{j+1} - (i-j) \sum_{l=1}^{[j/2]} \frac{1}{l} \begin{pmatrix} j-l \\ l-1 \end{pmatrix} \begin{pmatrix} i-l \\ j-2l \end{pmatrix} \left(\frac{(c+\lambda)}{\varepsilon\,(n+1)^2} \right)^{j-2l+1} \tag{3.27}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, i$.

3.5 NUMERICAL RESULTS

In the nonlinear case we need at least the uniform boundedness of sequences $\{\alpha_i^n(u)\}_{i=1}^n \subset \mathbb{R}$ as $n \to \infty$ for all u from some reasonable function space. We used Mathematica and MuPAD to calculate $\alpha_n^n(u)$ symbolically using the recursive relationship (3.18) and then to evaluate it for several different functions $u(x)$ and for different nonlinear functions $f(u)$. Since we found no qualitative difference between results corresponding to functions $u(x)$ of

[2] M grows with c, λ and $1/\varepsilon$.

the same size, we present here only the results for functions of the form $u(x) = p\sin(\pi x)$ with different values of p. The symbolic calculation becomes extremely demanding computationally for increasing values of n. We were able to evaluate α_n^n for values up to $n = 9$ or $n = 10$ depending on the complexity of the nonlinear function $f(u)$. The results are collected below in two tables.

1. In the case of $f(u) = u\ln\left(1+u^2\right)$ we have superlinearity $\frac{f(u)}{u} \xrightarrow{u\to\infty} \infty$, but the condition $\int_b^\infty \frac{du}{f(u)} < \infty$, which is necessary for finite time blow up (see, e.g. [9],) is not satisfied for any $b > 0$. Also, the zero solution of equation (3.1) is locally stable. The value $p = 1.5$ corresponds to an initial value for which the open–loop solution converges to zero. As the corresponding column in the table below shows the control operator α_n^n converges to a finite value. For $p = 2$ the uncontrolled solution of (3.1) does not converge to zero, but still α_n^n converges to a finite value. For larger values of p the convergence is not obvious from the calculations, but the concavity of the function graphs (decreasing rates of change in the values of α_n^n) suggest that we have convergence for increasing values of n with a decreasing rate of convergence as the size of the initial function is increased.

α_n^n for $f(u) = u\ln\left(1+u^2\right)$				
n	$p = 1.5$	$p = 2$	$p = 5$	$p = 10$
1	-4.4	-8.0	-40.7	-115.3
2	-4.5	-11.0	-97.4	-356.2
3	-4.4	-11.6	-141.1	-615.1
4	-4.3	-12.3	-178.4	-867.1
5	-4.3	-12.6	-209.0	-1099.1
6	-4.2	-12.8	-233.4	-1301.5
7	-4.2	-13.0	-252.5	-1472.6
8	-4.2	-13.1	-267.6	-1615.4
9	-4.2	-13.2	-279.5	-1733.6

1. For $f(u) = u^2$ solutions corresponding to large initial data exhibit finite time blow–up. In fact, all of the present p values correspond to initial functions that result in finite time blow–up. However, for $p = 1.5$ and $p = 2$, the control values seem to converge as the table below shows. For larger values ($p = 5$ and $p = 10$) numerical calculations suggest fast divergence.

α_n^n for $f(u) = u^2$				
n	$p = 1.5$	$p = 2$	$p = 5$	$p = 10$
1	−5.6	−10.0	−62.5	−250.0
2	−7.2	−16.1	−221.3	−1687.0
3	−7.6	−18.6	−402.0	−4974.2
4	−8.0	−21.1	−637.3	−11202.1
5	−8.2	−22.6	−926.7	−22798.3
6	−8.3	−23.8	−1244.8	−41999.6
7	−8.3	−24.6	−1578.1	−70862.2
8	−8.4	−25.3	−1915.4	−111498.4
9	−8.4	−25.8	−2247.4	−165709.2
10	−8.5	−26.1	−2567.5	−234811.7

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