Boundary Control of the Korteweg–de Vries–Burgers Equation: Further Results on Stabilization and Well Posedness, with Numerical Demonstration*

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Abstract

We consider the Korteweg–de Vries–Burgers (KdVB) equation on the interval $[0,1]$. Motivated by poor decay rates of a recently proposed control laws by Liu and Krstic which keeps some of the boundary conditions as homogeneous, we propose a strengthened set of feedback boundary conditions. We establish stability properties of the closed–loop system and illustrate the performance improvement by a simulation example.

***Keywords* —** Korteweg–de Vries–Burgers equation, nonlinear boundary feedback control, global stabilization.

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1 Introduction

The Korteweg–de Vries–Burgers (KdVB) equation is one of the simplest nonlinear mathematical models displaying the features of both dispersion and dissipation. It serves as a model of long waves in shallow water and some other physical phenomena. The usual and simplest setting in which the controlled and uncontrolled KdVB equation or the simpler KdV equation is considered is either the case of periodic boundary conditions (see, e.g. Biler [4], Bona et al [5, 6], Russel and Zhang [19]) or the case where the spatial domain is the whole real line (see, e.g. Biler [3], Bona and Smith [7]). As a next step in the analysis of a system it is natural to consider the controllability (see, e.g. Rosier [17]) and stabilization (see, e.g. Zhang [22]) on a bounded domain. In a recent work Liu and Krstic [15] consider a boundary feedback stabilization problem for a KdVB equation on a finite spatial interval. Our paper is motivated by relatively poor performance of the controllers in [15] which we have observed in numerical simulations. In this paper we propose a more aggressive control law that achieves better performance. Our control law can be implemented via any of the following three variables actuated at one boundary with w held at zero at the other boundary: (w_x, w_{xx}), (w, w_x), (w, w_{xx}). The uncontrolled versions of some of these problems are known not to be asymptotically stable. An example of a physical problem where our control law would be implementable is the water channel setup with boundary actuation discussed in Rosier [17]. In Section 2 we prove the existence and stability of solutions of the resulting boundary controlled KdVB equation. In Section 3 we provide a numerical example after a brief description of the finite difference numerical method we used.

2 Stabilization

Consider the Korteweg–de Vries–Burgers equation

$$w_t - \varepsilon w_{xx} + \delta w_{xxx} + w w_x = 0, \quad x \in [0,1], \quad t > 0, \tag{2.1}$$

with ε, $\delta > 0$ and with some initial data

$$w(x,0) = w_0(x), \quad x \in [0,1]. \tag{2.2}$$

Liu and Krstic [15] proposed the control law

$$\begin{aligned} w(0,t) &= 0, && (2.3)\\ w_x(1,t) &= 0, && (2.4)\\ w_{xx}(1,t) &= -\frac{1}{3\delta} w^2(1,t) && (2.5) \end{aligned}$$

which achieves global stability in L^2, and an improved version of it

$$\begin{aligned} w(0,t) &= 0, && (2.6)\\ w_x(1,t) &= 0, && (2.7)\\ w_{xx}(1,t) &= \frac{1}{\delta}\left(c + \frac{1}{9c} w^2(1,t)\right) w(1,t), \qquad c > 0 && (2.8) \end{aligned}$$

which achieves also global H^1–stability and guarantees well–posedness. Unfortunately, as we shall see in Section 3, the choice $w_x(1,t)=0$ results in slow convergence to zero. For this reason, in this paper we seek and find a more aggressive boundary condition that also uses $w_x(1,t)$ for feedback:

$$\begin{aligned} w(0,t) &= 0, &(2.9)\\ w_x(1,t) &= -g_1(w(1,t)) \triangleq -\frac{1}{\varepsilon}\left(c+\frac{1}{9c}w^2(1,t)\right)w(1,t), &(2.10)\\ w_{xx}(1,t) &= g_2(w(1,t)) \triangleq \frac{1}{\varepsilon^2}\left(c+\frac{1}{9c}w^2(1,t)\right)^2 w(1,t). &(2.11)\end{aligned}$$

It is clear that, since (2.10) and (2.11) are invertible functions, this control law can be implemented via any of the following three variables at the 1–boundary: (w_x, w_{xx}), (w, w_x), (w, w_{xx}).

In order to formulate our problem as an abstract initial value problem we consider Hilbert spaces $X=L^2(0,1), H=H^1(0,1)$, operator $\mathcal{A}:(\mathcal{D}(\mathcal{A})\subset X)\to X^*$given by

$$\mathcal{A}w = -\varepsilon w_{xx} + \delta w_{xxx} + \frac{1}{2}\left(w^2\right)_x, \tag{2.12}$$

and domain

$$\mathcal{D}(\mathcal{A}) = \left\{w\in H^3(0,1)\,|\,w(0)=0,\, w'(1)=-g_1(w(1)),\, w''(1)=g_2(w(1))\right\}. \tag{2.13}$$

With the above notation our system (2.1), (2.2), (2.9)–(2.11) can be written in the form of

$$\begin{aligned}\frac{dw}{dt}+\mathcal{A}w &= 0,\\ w(0) &= w_0. &(2.14)\end{aligned}$$

Our main result is formulated in the following theorem.

Theorem 1. *For any initial data $w_0\in\mathcal{D}(\mathcal{A})$ system (2.1), (2.2), (2.9)–(2.11) has a unique solution $w(x,t)\in C\left(0,\infty;L^2(0,1)\right)\cap C\left(0,\infty;H^1(0,1)\right)$ with*

1. *Global exponential stability in the L^2–sense:*

$$\|w(t)\| \le \|w_0\|e^{-\varepsilon t}, \qquad \forall t\ge 0, \tag{2.15}$$

2. *Global asymptotic and semi–global exponential stability in the H^1–sense: there exist $M>0$ such that for any $0\le\alpha<1$*

$$\|w(t)\|_{H^1} \le \frac{M}{\sqrt{1-\alpha}}\|w_0\|_{H^1} e^{\frac{M}{\sqrt{1-\alpha}}\|w_0\|_{H^1}^2} e^{-\alpha\varepsilon t}, \qquad \forall t\ge 0. \tag{2.16}$$

The same stability statements (reported in [15]) hold with control law (2.6)–(2.8). Since (2.15) and (2.16) are conservative energy estimates and M is a generic constant, they do not provide a good basis for comparison of the two controllers.

It is very important to understand the role of the parameter α in the H^1 estimate (2.16)[1]. The larger α, the better the exponential decay rate. However, at the same time, the "overshoot", which is proportional to $1/\sqrt{1-\alpha}$, grows and approaches infinity as $\alpha\to 1$.

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[1] We thank a reviewer and associate editor for suggesting the use of this parameter. Our original analysis was for $\alpha=1/2$.

Proof of Theorem 1. In order to prove the stability results we use energy estimates. First take the L^2–inner product of (2.1) with w to obtain

$$\int_0^1 w_t w\,dx - \varepsilon \int_0^1 w_{xx} w\,dx + \delta \int_0^1 w_{xxx} w\,dx + \int_0^1 w_x w^2\,dx = 0\,. \tag{2.17}$$

Using mainly integration by parts, we can write the various terms as

$$\int_0^1 w_t w\,dx = \frac{1}{2}\frac{d}{dt}\|w(t)\|^2\,, \tag{2.18}$$

$$\begin{aligned} -\varepsilon \int_0^1 w_{xx} w\,dx &= -\varepsilon w_x w|_0^1 + \varepsilon \|w_x(t)\|^2 \\ &= -\varepsilon w_x(1,t)\,w(1,t) + \varepsilon \|w_x(t)\|^2 \\ &= \left(c + \frac{1}{9c} w^2(1,t)\right) w^2(1,t) + \varepsilon \|w_x(t)\|^2\,, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \delta \int_0^1 w_{xxx} w\,dx &= \delta w_{xx} w|_0^1 - \delta \int_0^1 w_{xx} w_x\,dx \\ &= \frac{\delta}{\varepsilon^2}\left(c + \frac{1}{9c} w^2(1,t)\right)^2 w^2(1,t) - \frac{\delta}{2} w_x^2|_0^1 \\ &= \frac{\delta}{2\varepsilon^2}\left(c + \frac{1}{9c} w^2(1,t)\right)^2 w^2(1,t) + \frac{\delta}{2} w_x^2(0,t)\,, \end{aligned} \tag{2.20}$$

and

$$\int_0^1 w_x w^2\,dx = \frac{1}{3} w^3(1,t) \leq \frac{1}{18c} w^4(1,t) + \frac{c}{2} w^2(1,t)\,. \tag{2.21}$$

Substituting (2.18)–(2.21) into (2.17) and simplifying the resulting inequality we obtain

$$\frac{d}{dt}\|w(t)\|^2 + 2\varepsilon\|w_x\|^2 + \left(\frac{\delta c^2}{\varepsilon^2} + c\right) w^2(1,t) + \left(\frac{1}{9c} + \frac{2\delta}{9\varepsilon^2}\right) w^4(1,t) + \frac{\delta}{81c^2\varepsilon^2} w^6(1,t) \leq 0\,. \tag{2.22}$$

As a first consequence of (2.22) we obtain, using Poincaré's inequality, the inequality

$$\frac{d}{dt}\|w(t)\|^2 \leq -2\varepsilon\|w_x(t)\|^2 \leq -2\varepsilon\|w(t)\|^2\,, \tag{2.23}$$

which implies (2.15), i.e. the global exponential stability in the L^2 sense:

$$\|w(t)\| \leq \|w_0\|\,e^{-\varepsilon t}\,. \tag{2.24}$$

We now return back to (2.22) and multiply it by $e^{2\alpha\varepsilon t}$, where $0 \leq \alpha < 1$ is arbitrary. Using (2.24) we get

$$\begin{aligned} &\frac{d}{dt}\left(e^{2\alpha\varepsilon t}\|w(t)\|^2\right) + 2\varepsilon e^{2\alpha\varepsilon t}\|w_x(t)\|^2 + e^{2\alpha\varepsilon t}\left(\frac{\delta c^2}{\varepsilon} + c\right) w^2(1,t) \\ &+ e^{2\alpha\varepsilon t}\left(\frac{1}{9c} + \frac{2\delta}{9\varepsilon^2}\right) w^4(1,t) + \frac{\delta e^{2\alpha\varepsilon t}}{81c^2\varepsilon^2} w^6(1,t) \leq 2\alpha\varepsilon e^{2\alpha\varepsilon t}\|w(t)\|^2 \\ &\leq 2\alpha\varepsilon\|w_0\|^2 e^{-(2-2\alpha)\varepsilon t}\,. \end{aligned} \tag{2.25}$$

Integrating (2.25) with respect to time we obtain

$$e^{2\alpha\varepsilon t}\|w(t)\|^2+\int_0^t e^{2\alpha\varepsilon\tau}\left(\|w_x(\tau)\|^2+w^2(1,\tau)+w^4(1,\tau)+w^6(1,\tau)\right)d\tau\le\frac{1}{1-\alpha}\|w_0\|^2. \tag{2.26}$$

Next, we take the L^2–inner product of (2.1) with $-w_{xx}$ to obtain

$$-\int_0^1 w_t w_{xx}\,dx+\varepsilon\|w_{xx}\|^2-\delta\int_0^1 w_{xxx}w_{xx}\,dx-\int_0^1 ww_xw_{xx}\,dx=0. \tag{2.27}$$

The various terms of (2.27) can be written in the following way.

$$\begin{aligned}-\int_0^1 w_tw_{xx}\,dx &= -w_tw_x|_0^1+\frac{1}{2}\frac{d}{dt}\|w_x\|^2\\ &= -w_t(1,t)\,w_x(1,t)+\frac{1}{2}\frac{d}{dt}\|w_x\|^2\\ &= \frac{1}{\varepsilon}w_t(1,t)\left(c+\frac{1}{9c}w^2(1,t)\right)w(1,t)+\frac{1}{2}\frac{d}{dt}\|w_x\|^2\\ &= \frac{1}{2}\frac{d}{dt}\left(\|w_x\|^2+\frac{c}{\varepsilon}w^2(1,t)+\frac{1}{18\varepsilon c}w^4(1,t)\right),\end{aligned} \tag{2.28}$$

$$\delta\int_0^1 w_{xxx}w_{xx}\,dx=\frac{\delta}{2}w_{xx}^2|_0^1=\frac{\delta}{2\varepsilon^4}\left(c+\frac{1}{9c}w^2(1,t)\right)^4w^2(1,t)-\frac{\delta}{2}w_{xx}^2(0,t), \tag{2.29}$$

$$\begin{aligned}\int_0^1 w_xww_{xx}\,dx &\le \|w(t)\|_{L^\infty}\int_0^1|w_xw_{xx}|\,dx\\ &\le \|w(t)\|_{L^\infty}\|w_x(t)\|\,\|w_{xx}(t)\|\\ &\le \frac{1}{2\varepsilon}\|w(t)\|_{L^\infty}^2\|w_x(t)\|^2+\frac{\varepsilon}{2}\|w_{xx}(t)\|^2\\ &\le \frac{1}{2\varepsilon}\|w_x(t)\|^4+\frac{\varepsilon}{2}\|w_{xx}(t)\|^2.\end{aligned} \tag{2.30}$$

Here, in the last step, we used the simple inequality $\|w\|_{L^\infty(0,1)}\le\|w_x\|$, which holds for $w\in H_0^1(0,1)$. Introducing the notation

$$A(t)\equiv\frac{c}{\varepsilon}w^2(1,t)+\frac{1}{18\varepsilon c}w^4(1,t)+\|w_x(t)\|^2 \tag{2.31}$$

and substituting (2.28)–(2.30) into (2.27) we obtain

$$\begin{aligned}\frac{d}{dt}A(t)+\varepsilon\|w_{xx}(t)\|^2 &\le \frac{\delta}{\varepsilon^4}\left(c^2w^2(1,t)+\frac{2}{9}w^4(1,t)+\frac{1}{18c^2}w^6(1,t)\right)\\ &\quad\times\left(c^2+\frac{2}{9}w^2(1,t)+\frac{1}{81c^2}w^4(1,t)\right)-\delta w_{xx}^2(0,t)+\frac{1}{\varepsilon}\|w_x(t)\|^4\\ &\le M\left(w^2(1,t)+w^4(1,t)+w^6(1,t)+\|w_x(t)\|^2\right)\\ &\quad+M\left(w^2(1,t)+w^4(1,t)+w^6(1,t)+\|w_x(t)\|^2\right)A(t).\end{aligned} \tag{2.32}$$

Omitting the nonnegative second term on the left, using definitions (2.31) and

$$b(t) \equiv e^{2\alpha\varepsilon t}\left(w^2(1,t) + w^4(1,t) + w^6(1,t) + \|w_x(t)\|^2\right), \tag{2.33}$$

and multiplying (2.32) by $e^{2\alpha\varepsilon t}$ we get

$$\frac{d}{dt}\left(e^{2\alpha\varepsilon t}A(t)\right) \leq Mb(t) + Mb(t)\,e^{2\alpha\varepsilon t}A(t). \tag{2.34}$$

After integration we obtain from here

$$e^{2\alpha\varepsilon t}A(t) \leq A(0) + \int_0^t Mb(\tau)\,d\tau + \int_0^t Mb(\tau)\,e^{2\alpha\varepsilon\tau}A(\tau)\,d\tau. \tag{2.35}$$

It follows now from Gronwall's inequality, estimate (2.26) and the definition of $b(t)$ that

$$\begin{aligned} e^{2\alpha\varepsilon t}A(t) &\leq \left(A(0) + \int_0^t Mb(\tau)\,d\tau\right)\left(1 + \int_0^t Mb(\tau)\exp\left(\int_\tau^t Mb(s)\,ds\right)d\tau\right) \\ &\leq \left(A(0) + \frac{M\|w_0\|^2}{1-\alpha}\right) + \left(A(0) + \frac{M\|w_0\|^2}{1-\alpha}\right)\frac{M\|w_0\|^2}{1-\alpha}e^{\frac{M\|w_0\|^2}{1-\alpha}}. \end{aligned} \tag{2.36}$$

Multiplying (2.36) by $e^{-2\alpha\varepsilon t}$, taking the square root, and using the definition of $A(t)$ one more time we arrive at the inequality

$$\|w(t)\|_{H^1} \leq k(\alpha)\,\|w_0\|_{H^1}\,e^{k(\alpha)\|w_0\|^2_{H^1}}e^{-\alpha\varepsilon t}, \tag{2.37}$$

where

$$k(\alpha) = \frac{M}{\sqrt{1-\alpha}}. \tag{2.38}$$

This proves (2.16), the semi–global exponential stability in the H^1–sense. Notice that $k(\alpha)$ is a monotone increasing function of its argument α and it blows up at $\alpha = 1$. The decay rate dependent overshoot coefficient dominates the estimate on short time intervals, which shows again the need for numerical comparison. Due to the general Sobolev embedding theorem $H^\ell(\Omega) \subset C^k\left(\overline{\Omega}\right)$, which holds for $k \leq \ell - \frac{n}{2}$, $\Omega \subset \mathbb{R}^n$, the solution $w(t,x)$ is continuous and bounded for all $t \geq 0$ and all $x \in [0,1]$.

For completeness we include here an existence proof that is based on the theory of monotone operators with locally Lipschitz perturbations [9, 20, 1] and follows the arguments in [14][2].

We consider two operators, $\mathcal{A}_1 : (\mathcal{D}(\mathcal{A}_1) \subset X) \to X$ given by

$$\mathcal{A}_1 w = -\varepsilon_1 w_{xx} + \delta w_{xxx} \tag{2.39}$$

with domain

$$\mathcal{D}(\mathcal{A}_1) = \left\{w \in H^3(0,1) \mid w(0) = 0,\, w'(1) = -g_1(w(1)),\, w''(1) = g_2(w(1))\right\}, \tag{2.40}$$

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[2] See Acknowledgment section at the end of the paper.

and $\mathcal{A}_2 : (\mathcal{D}(\mathcal{A}_2) \subset X) \to X$ given by

$$\mathcal{A}_2 w = \lambda w - \varepsilon_2 w_{xx} + \frac{d}{dx} f(w) \tag{2.41}$$

with domain

$$\mathcal{D}(\mathcal{A}_2) = \left\{ w \in H^2(0,1) \mid w(0) = 0, w'(1) = -g_1(w(1)) \right\} . \tag{2.42}$$

Here $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $f(w) = w^2/2$.

Next we introduce a cut–off function of $f(y) = y^2/2$ and obtain the globally Lipschitz continuous function

$$f_K(y) = \begin{cases} y^2/2 & \text{if } |y| \leq K, \\ K^2/2 & \text{if } |y| > K \end{cases} \tag{2.43}$$

with Lipschitz constant $L_K = K$. We define the nonlinear operator $\mathcal{A}_{2,K}$ corresponding to the cut–off version of $\mathcal{A}_2$ as

$$\mathcal{A}_{2,K} w = \lambda_K I - \varepsilon_2 w_{xx} + \frac{d}{dx} f_K(w) \tag{2.44}$$

for some $\lambda_K \in \mathbb{R}$ with domain $\mathcal{D}(\mathcal{A}_{2,K}) = \mathcal{D}(\mathcal{A}_2)$.

Our plan is to consider the abstract, truncated Cauchy problem

$$\begin{aligned} \frac{dw}{dt} + (\mathcal{A}_K - \lambda_K I) w &= 0, \qquad t > 0, && (2.45) \\ w(0) &= 0, && (2.46) \end{aligned}$$

where $\mathcal{A}_K w = \mathcal{A}_1 w + \mathcal{A}_{2,K} w = \lambda_K w - \varepsilon w_{xx} + \delta w_{xxx} + \frac{d}{dx} f_K(w)$. We will show that problem (2.45)–(2.46) has a strong solution w_K for all $K > 0$ and then we obtain a variational solution of the original problem as the limit of w_K, in an appropriate sense, as $K \to \infty$.

We have to show that $\mathcal{A}_K$ is m–accretive (maximal monotone or, with other words $-\mathcal{A}_K$ is maximal dissipative) on X in order to use the Crandall–Liggett Theorem. First we show the monotonicity of $\mathcal{A}_K$ for some $\lambda_K \in \mathbb{R}$ by showing the monotonicity of $\mathcal{A}_1$ and $\mathcal{A}_{2,K}$ separately.

$$\begin{aligned}
&\langle \mathcal{A}_1 w - \mathcal{A}_1 v, w - v \rangle \\
&= -\varepsilon_1 \langle (w-v)_{xx}, w - v \rangle + \delta \langle (w-v)_{xxx}, w - v \rangle \\
&= -\varepsilon_1 (w-v)_x (w-v) |_0^1 + \varepsilon_1 \|(w-v)_x\|^2 + \delta (w-v)_{xx} (w-v) |_0^1 \\
&\quad - \delta \langle (w-v)_{xx}, (w-v)_x \rangle \\
&= \varepsilon_1 (g_1(w(1)) - g_1(v(1))) (w(1) - v(1)) + \varepsilon_1 \|w_x - v_x\|^2 \\
&\quad + \delta (g_2(w(1)) - g_2(v(1))) (w(1) - v(1)) - \frac{\delta}{2} (w_x - v_x)^2 |_0^1 \\
&= \varepsilon_1 (g_1(w(1)) - g_1(v(1))) (w(1) - v(1)) + \varepsilon_1 \|w_x - v_x\|^2 \\
&\quad + \delta (g_2(w(1)) - g_2(v(1))) (w(1) - v(1)) \\
&\quad + \frac{\delta}{2} (g_1(w(1)) - g_1(v(1)))^2 + \frac{\delta}{2} (w_x(0) - v_x(0))^2 \\
&\geq \varepsilon_1 \|w_x - v_x\|^2 + \frac{\delta}{2} g_2(w(1)) w(1) + \frac{\delta}{2} w_x^2(0) \\
&\geq \varepsilon_1 \|w_x - v_x\|^2 \geq \varepsilon_1 \|w - v\|^2 ,
\end{aligned} \tag{2.47}$$

where we have used the fact that both g_1 and g_2 are monotone functions and in the first inequality we have also used their explicit form. As a result we obtain that the operator $\mathcal{A}_1$ is monotone on X and on $H \times H^*$. It is also maximal monotone, since its restriction to homogeneous boundary conditions is a linear, maximal monotone operator.

$$\begin{aligned}
&\langle \mathcal{A}_{2,K} w - \mathcal{A}_{2,K} v, w - v \rangle \\
&= \lambda_K \|w - v\|^2 - \varepsilon_2 \langle (w-v)_{xx}, w - v \rangle + \langle (f_K(w))_x - (f_K(v))_x, w - v \rangle \\
&= \lambda_K \|w - v\|^2 - \varepsilon_2 (w-v)_x (w-v) |_0^1 + \varepsilon_2 \|(w-v)_x\|^2 \\
&\quad + (f_K(w) - f_K(v))(w - v) |_0^1 - \langle f_K(w) - f_K(v), (w-v)_x \rangle \\
&\geq \lambda_K \|w - v\|^2 + \varepsilon_2 (g_1(w(1)) - g_1(v(1)))(w(1) - v(1)) + \varepsilon_2 \|w_x - v_x\|^2 \\
&\quad + (f_K(w(1)) - f_K(v(1)))(w(1) - v(1)) - \|f_K(w) - f_K(v)\| \|w_x - v_x\| \\
&\geq \lambda_K \|w - v\|^2 + \varepsilon_2 (g_1(w(1)) - g_1(v(1)))(w(1) - v(1)) + \varepsilon_2 \|w_x - v_x\|^2 \\
&\quad - K(w(1) - v(1))^2 - \frac{\gamma}{2} \|w_x - v_x\|^2 - \frac{1}{2\gamma} \|f_K(w) - f_K(v)\| \\
&\geq \lambda_K \|w - v\|^2 + \varepsilon_2 \|w_x - v_x\|^2 - \frac{\gamma}{2} \|w_x - v_x\|^2 - \frac{K}{2\gamma} \|w - v\| \\
&\quad - 2K \|w - v\| \|w_x - v_x\| \\
&\geq \left(\lambda_K - \frac{K}{2\gamma} - \frac{2K^2}{\gamma} \right) \|w - v\|^2 + (\varepsilon_2 - \gamma) \|w_x - v_x\|^2 . \qquad (2.48)
\end{aligned}$$

Here we used integration by parts in the first step, the boundary conditions and the Cauchy–Schwarz inequality in the second step. In the remaining steps we exploited the Lipschitz continuity of f_K, along with Young's inequality. We also used the simple inequality $\left|\varphi^2(1)\right| \leq 2\|\varphi_x\|\|\varphi\|$. The term containing g_1 is positive as g_1 is a monotone function, and hence this term was dropped from the last two estimates. Choosing $\gamma = \varepsilon_2/2$ in (2.48) we obtain that $\mathcal{A}_{2,K}$ is monotone both on X and on $H \times H^*$ for λ_K large enough (for example for $\lambda_K \geq 2K(4K+1)/\varepsilon_2$). It also follows that $\mathcal{A}_{2,K}$ is hemicontinuous. Putting together $\mathcal{A}_1$ and $\mathcal{A}_{2,K}$ we obtain that $\mathcal{A}_K = \mathcal{A}_1 + \mathcal{A}_{2,K}$ is monotone on X and $H \times H^*$and hemicontinuous on $H \times H^*$ for large ω_K. The operator $\mathcal{A}_K$ is also coercive on $H \times H^*$ since (2.47) and (2.48) imply (with $v = 0$) that

$$\lim_{\|w\| \to \infty} \frac{\langle \mathcal{A}_K w, w \rangle}{\|w\|_{H^1}} \geq \frac{\langle \mathcal{A}_{2,K} w, w \rangle}{\|w\|_{H^1}} \geq \lim_{\|v\| \to \infty} \frac{\frac{\varepsilon_2}{2} \|w\|_{H^1}^2}{\|w\|_{H^1}} = \infty .$$

As a result, by [1, Corollary 1.3, page 46], the operator $\mathcal{A}_K : H \to H^*$ (as well as $\mathcal{A}_K - \lambda I$) is surjective. Due to this result and the inclusion $H \subset X \subset H^*$, in order to show that the range of $\mathcal{A}_K - \lambda I$ is all of X, it suffices to show that if $f \in X$ and $w \in H$ satisfies

$$\mathcal{A}_K w - \lambda w = f \qquad (2.49)$$

then $w \in \mathcal{D}(\mathcal{A}_K)$. Expanding and rearranging (2.49) we get

$$-\varepsilon w_{xx} + \delta w_{xxx} = f + (\lambda - \lambda_K) w - \frac{d}{dx} f_K(w) \in X \qquad (2.50)$$

and since we already know that $\mathcal{A}_1$ is maximal monotone for all $\varepsilon_1, \delta > 0$, we obtain that $w \in \mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}_K)$. Hence, by Minty's Theorem [16] $\mathcal{A}_K$ is maximal monotone on X and by the Crandall–Liggett Theorem [9] problem (2.45)–(2.46) has a unique strong solution $w_K \in C(0,\infty;\mathcal{D}(\mathcal{A}_K)) \cap C^1\left(0,\infty;L^2(0,1)\right) \subset C\left(0,\infty;H^3(0,1)\right) \cap C^1\left(0,\infty;L^2(0,1)\right)$ for all $K>0$.

Next, we establish the uniform boundedness of the sequence $\{w_K\}_{K>0}$ in the same way as the a priori estimates were obtained. Starting with the identity

$$\int_0^1 w_{Kt}w_K\,dx - \varepsilon\int_0^1 w_{Kxx}w_K\,dx + \delta\int_0^1 w_{Kxxx}w_K\,dx + \int_0^1 \left(f_K(w_K)\right)_x w_K\,dx = 0 \tag{2.51}$$

we estimate the last term as

$$\int_0^1 \left(f_K(w_K)\right)_x w_K\,dx = \int_{|w_K|<K} \left(f_K(w_K)\right)_x w_K\,dx = \frac{1}{3}\int_{|w_K|<K} \left(w_K^3\right)_x dx \le \frac{1}{3}\left|w_K(1,t)\right|^3, \tag{2.52}$$

and with this we obtain the uniform in K a priori estimate (2.26) for w_K

$$e^{2\alpha\varepsilon t}\left\|w_K(t)\right\|^2 + \int_0^t e^{2\alpha\varepsilon\tau}\left(\left\|w_{Kx}(\tau)\right\|^2 + w_K^2(1,\tau) + w_K^4(1,\tau) + w_K^6(1,\tau)\right)d\tau \le \frac{1}{1-\alpha}\left\|w_0\right\|^2. \tag{2.53}$$

We obtain estimate (2.16) similarly, after noting that estimate (2.30) is now written as

$$\begin{aligned}
\int_0^1 \left(f_K(w_K)\right)_x w_{Kxx}\,dx &= \int_{|w_K|<K} \left(f_K(w_K)\right)_x w_{Kxx}\,dx \\
&= \int_{|w_K|<K} w_{Kx}w_K w_{Kxx}\,dx \\
&\le \int_0^1 \left|w_{Kx}w_K w_{Kxx}\right| dx \\
&\le \frac{1}{2\varepsilon}\left\|w_{Kx}(t)\right\|^4 + \frac{\varepsilon}{2}\left\|w_{Kxx}(t)\right\|^2. \qquad (2.54)
\end{aligned}$$

Consider now two parameters K, L and the corresponding two solutions w_K, w_L of (2.45)–(2.46). For their difference $w = w_K - w_L$ we have

$$w_t - \varepsilon w_{xx} + \delta w_{xxx} + \left(f_K(w_K)\right)_x - \left(f_L(w_L)\right)_x = 0, \qquad x\in[0,1], \quad t>0, \tag{2.55}$$

$$\begin{aligned}
w(x,0) &= 0, & (2.56)\\
w(0,t) &= 0, & (2.57)\\
w_x(1,t) &= g_1(w_L(1,t)) - g_1(w_K(1,t)) = w(1,t)\,\tilde{g}_1, & (2.58)\\
w_{xx}(1,t) &= g_2(w_K(1,t)) - g_2(w_L(1,t)) = w(1,t)\,\tilde{g}_2, & (2.59)
\end{aligned}$$

where

$$\tilde{g}_1 = -\frac{c}{9\varepsilon}\left(9 + w_K^2(1,t) + w_K(1,t)\,w_L(1,t) + w_L^2(1,t)\right) \tag{2.60}$$

and

$$\tilde{g}_2 = \frac{c^2}{81\varepsilon^2}\left(\left(9+w_L^2(1,t)\right)^2 + w_K(1,t)\left(w_K(1,t)+w_L(1,t)\right)\left(w_K^2(1,t)+18+w_L^2\right)\right). \tag{2.61}$$

After taking the inner product of (2.55) with w we calculate

$$\begin{aligned}
\varepsilon\int_0^1 w_{xx}w\,dx &= \varepsilon w_x w|_0^1 - \varepsilon\left\|w_x(t)\right\|^2 \\
&= \varepsilon\tilde{g}_1 w^2(1,t) - \varepsilon\left\|w_x(t)\right\|^2 \\
&\le C\left|\tilde{g}_1\right|\|w\|\|w_x\| - \varepsilon\|w_x\|^2 \\
&\le C\tilde{g}_1^2\|w\|^2 - \frac{3\varepsilon}{4}\|w_x\|^2
\end{aligned} \tag{2.62}$$

$$\begin{aligned}
&\delta\int_0^1 w_{xxx}w\,dx = \delta w_{xx}w|_0^1 - \delta\int_0^1 w_{xx}w_x\,dx \\
&= \delta\tilde{g}_2 w^2(1,t) - \frac{\delta}{2}w_x^2|_0^1 \\
&= \delta\tilde{g}_2 w^2(1,t) - \frac{\delta}{2}\tilde{g}_1^2 w^2(1,t) + \frac{\delta}{2}w_x^2(0,t) \\
&= \frac{\delta c^2}{162\varepsilon^2}w^2(1,t)\left(81 + 18\left(w_K^2(1,t) + w_L^2(1,t)\right) + \left(w_K^2(1,t) - w_L^2(1,t)\right)^2\right) \\
&+\frac{\delta c^2}{162\varepsilon^2}w^2(1,t)\,w_K^2(1,t)\,w_L^2(1,t) + \frac{\delta}{2}w_x^2(0,t) \\
&-\frac{\delta c^2}{9\varepsilon^2}w^2(1,t)\,w_K(1,t)\,w_L(1,t)\,.
\end{aligned} \tag{2.63}$$

Here only the last term is not positive definite and it can be estimated as

$$\begin{aligned}
&\frac{\delta c^2}{9\varepsilon^2}w^2(1,t)\,w_K(1,t)\,w_L(1,t) \le C\|w\|\|w_x\|\left(\|w_K\| + \|w_L\|\right)\left(\|w_{Kx}\| + \|w_{Lx}\|\right) \\
&\le C\left(\|w_K\| + \|w_L\|\right)^2\left(\|w_{Kx}\| + \|w_{Lx}\|\right)^2\|w\|^2 + \frac{\varepsilon}{4}\|w_x\|^2\,.
\end{aligned} \tag{2.64}$$

Then, using the notation

$$L_1(t) = C\left(\tilde{g}_1^2 + \left(\|w_K\| + \|w_L\|\right)^2\left(\|w_{Kx}\| + \|w_{Lx}\|\right)^2 + \|w_{Kx}\|^2 + \|w_L\|\|w_{Lx}\|\right) \tag{2.65}$$

we obtain

$$\frac{d}{dt}\|w(t)\|^2 + \varepsilon\|w_x(t)\|^2 \le L_1(t)\|w(t)\|^2 + \int_0^1\left(\left(f_K(w_K)\right)_x - \left(f_L(w_L)\right)_x\right)\left(w_K - w_L\right)dx. \tag{2.66}$$

From here, using Gronwall's inequality we obtain

$$\begin{aligned}
&\|w(t)\|^2 + \int_0^t\|w_x(\tau)\|^2\,d\tau \le \int_0^t\int_0^1\left(\left(f_K(w_K)\right)_x - \left(f_L(w_L)\right)_x\right)\left(w_K - w_L\right)dx\,d\tau \\
&\times\left(1 + \int_0^t L_1(\tau)\exp\left[\int_0^\tau L_1(s)\,ds\right]d\tau\right).
\end{aligned} \tag{2.67}$$

Since the first factor converges to zero according to Lemma 1 and the second factor is bounded according to (2.53), we obtain

$$\left\|\left(w_K - w_L\right)(t)\right\|^2 + \int_0^t\left\|w_{Kx}(\tau) - w_{Lx}(\tau)\right\|^2 d\tau \to 0, \qquad K,L\to\infty. \tag{2.68}$$

With (2.68) we obtain that

$$w_K \xrightarrow{K\to\infty} w \ \text{ in } C\left(0,T;L^2(0,1)\right)\cap L^2\left(0,T;H^1(0,1)\right). \tag{2.69}$$

Taking the inner product of (2.55) with $-w_{xx}$ we obtain

$$\begin{aligned} -\int_0^1 w_t w_{xx}\,dx &= -w_t w_x|_0^1 + \frac{1}{2}\frac{d}{dt}\|w_x\|^2 \\ &= -w_t(1,t)\,w_x(1,t) + \frac{1}{2}\frac{d}{dt}\|w_x\|^2 \\ &= -w_t(1,t)\,w(1,t)\,\tilde{g}_1 + \frac{1}{2}\frac{d}{dt}\|w_x\|^2 \\ &= G(w_K,w_L)\frac{d}{dt}\left(\|w_x\|^2 + w^2(1,t)\right), \end{aligned} \tag{2.70}$$

where $0 < G(w_K,w_L)$ depends also on the sign of $\frac{d}{dt}\|w_x\|^2$ and $\frac{d}{dt}w^2(1,t)$.

$$\begin{aligned} \delta\int_0^1 w_{xxx}w_{xx}\,dx &= \frac{\delta}{2}\,w_{xx}^2\Big|_0^1 \\ &= \frac{\delta}{2}w^2(1,t)\,\tilde{g}_2 - \frac{\delta}{2}w_{xx}^2(0,t) \end{aligned} \tag{2.71}$$

We obtain

$$\begin{aligned} &G(w_K,w_L)\frac{d}{dt}\left(\|w_x(t)\|^2 + w^2(1,t)\right) \\ &\le \frac{\delta}{2}w^2(1,t)\,\tilde{g}_2 + \int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x\right) w_{xx}\,dx \end{aligned} \tag{2.72}$$

which can, in turn, be written as

$$\begin{aligned} \frac{d}{dt}\left(\|w_x(t)\|^2 + w^2(1,t)\right) &\le L_2(t)\left(\|w_x(t)\|^2 + w^2(1,t)\right) \\ &\quad + \int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x\right) w_{xx}\,dx. \end{aligned} \tag{2.73}$$

From here, using Gronwall's inequality we obtain

$$\begin{aligned} \|w_x(t)\|^2 + w^2(1,t) &\le \int_0^t\int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x\right) w_{xx}\,dx\,d\tau \\ &\quad \times\left(1 + \int_0^t L_2(\tau)\exp\left[\int_0^\tau L_2(s)\,ds\right]d\tau\right) \end{aligned} \tag{2.74}$$

Since the first factor converges to zero according to Lemma 2 and the second factor is bounded, we obtain

$$\|(w_{Kx} - w_{Lx})(t)\|^2 \to 0, \qquad \text{as } K,L\to\infty. \tag{2.75}$$

With this we obtain that

$$w_K \xrightarrow{K\to\infty} w \ \text{ in } C\left(0,T;L^2(0,1)\right)\cap C\left(0,T;H^1(0,1)\right), \tag{2.76}$$

where $w(x,t) \in C\left(0,T;L^2(0,1)\right) \cap C\left(0,T;H^1(0,1)\right)$ is a variational solution of problem (2.1), (2.2), (2.9)-(2.11) satisfying stability estimates (2.15) and (2.16).

The uniqueness is obtained taking two assumed solutions w_1 and w_2 and subtracting the corresponding two systems from each other. Then, using the notation $w = w_1 - w_2$ we obtain

$$w_t - \varepsilon w_{xx} + \delta w_{xxx} + w w_{1x} + w_2 w_x = 0, \qquad x \in [0,1], \quad t > 0, \tag{2.77}$$

$$\begin{aligned}
w(0,t) &= 0, \qquad &(2.78)\\
w_x(1,t) &= g_1(w_2(1,t)) - g_1(w_1(1,t)) = w(1,t)\tilde{g}_1, &(2.79)\\
w_{xx}(1,t) &= g_2(w_1(1,t)) - g_2(w_2(1,t)) = w(1,t)\tilde{g}_2, &(2.80)
\end{aligned}$$

where $\tilde{g}_1$ and $\tilde{g}_2$ have the same form as in (2.60)–(2.61) except w_K and w_L are now replaced by w_1 and w_2 respectively. We calculate

$$\begin{aligned}
\varepsilon \int_0^1 w_{xx} w \, dx &= \varepsilon w_x w|_0^1 - \varepsilon \|w_x(t)\|^2 \\
&= \varepsilon \tilde{g}_1 w^2(1,t) - \varepsilon \|w_x(t)\|^2 \\
&\leq C|\tilde{g}_1| \|w\| \|w_x\| - \varepsilon \|w_x\|^2 \\
&\leq C\tilde{g}_1^2 \|w\|^2 - \frac{7\varepsilon}{8}\|w_x\|^2
\end{aligned} \tag{2.81}$$

and

$$\begin{aligned}
\delta \int_0^1 w_{xxx} w \, dx &= \delta w_{xx} w|_0^1 - \delta \int_0^1 w_{xx} w_x \, dx \\
&= \delta \tilde{g}_2 w^2(1,t) - \frac{\delta}{2} w_x^2|_0^1 \\
&= \delta \tilde{g}_2 w^2(1,t) - \frac{\delta}{2}\tilde{g}_1^2 w^2(1,t) + \frac{\delta}{2} w_x^2(0,t) \\
&= \frac{\delta c^2}{162\varepsilon^2} w^2(1,t)\left(81 + 18\left(w_1^2(1,t) + w_2^2(1,t)\right) + \left(w_1^2(1,t) - w_2^2(1,t)\right)^2\right) \\
&\quad + \frac{\delta c^2}{162\varepsilon^2} w^2(1,t) w_1^2(1,t) w_2^2(1,t) + \frac{\delta}{2} w_x^2(0,t) \\
&\quad - \frac{\delta c^2}{9\varepsilon^2} w^2(1,t) w_1(1,t) w_2(1,t).
\end{aligned} \tag{2.82}$$

Here only the last term is not positive definite and it can be estimated as

$$\begin{aligned}
&\frac{\delta c^2}{9\varepsilon^2} w^2(1,t) w_1(1,t) w_2(1,t) \leq C\|w\| \|w_x\| \left(\|w_1\| + \|w_2\|\right)\left(\|w_{1x}\| + \|w_{2x}\|\right) \\
&\leq C\left(\|w_1\| + \|w_2\|\right)^2 \left(\|w_{1x}\| + \|w_{2x}\|\right)^2 \|w\|^2 + \frac{\varepsilon}{8}\|w_x\|^2
\end{aligned} \tag{2.83}$$

and

$$\int_0^1 w^2 w_{1x} \, dx \leq \left\|w^2\right\|_{L^\infty} \|w_{1x}\| \leq 2\|w\| \|w_x\| \|w_{1x}\| \leq C\|w_{1x}\|^2 \|w\|^2 + \frac{\varepsilon}{8}\|w_x\|^2. \tag{2.84}$$

Similarly

$$\int_0^1 w_2 w_x w\, dx \le C \|w_2\| \|w_{2x}\| \|w\|^2 + \frac{\varepsilon}{8} \|w_x\|^2 . \tag{2.85}$$

Then, using notation (2.65) with w_K and w_L are replaced by w_1 and w_2 we obtain

$$\frac{d}{dt} \|w(t)\|^2 \le L_1(t) \|w(t)\|^2 . \tag{2.86}$$

From here Gronwall's inequality implies

$$\|w(t)\|^2 \le \|w_0\|^2 \exp\left(\int_0^t L_1(\tau)\, d\tau \right). \tag{2.87}$$

Inspecting $L_1(t)$ we observe that it is integrable and since $\|w_0\| = 0$ we obtain that $\|w(t)\| = 0$ for all $t \ge 0$, i.e. the solution of (2.1), (2.2), (2.9)-(2.11) is unique.

□

Lemma 1. *Under the hypotheses of Theorem 1*

$$\int_0^t \int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x \right) (w_K - w_L)\, dx d\tau \xrightarrow{K,L \to \infty} 0 \tag{2.88}$$

for any $t > 0$.

Proof of Lemma 1. Let us use the notation $w = w_K - w_L$ and $\Omega_K = \{x \in [0,1] : |w_K(x)| > K\}$. The measure of Ω_K can be estimated as

$$\text{mes } \Omega_K = K^{-6} \int_{\Omega_K} K^6\, dx \le K^{-6} \|w_K\|_{L^6}^6 \le 4K^{-6} \|w_{Kx}\|^2 \|w_K\|^4 , \tag{2.89}$$

where we used the classical multiplicative inequality (2.99) with $r = m = 2$, and $q = 6$.

We have

$$\begin{aligned} &\int_0^t \int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x \right) (w_K - w_L)\, dx d\tau = \int_0^t \int_{\Omega_K \setminus \Omega_L} w_L w_{Lx} (w_L - w_K)\, dx d\tau \\ &\le 2 \left(\int_0^t \|w_{Lx}\|^2\, d\tau \right)^{1/2} \left(\int_0^t \int_{\Omega_K} |w_K|^4\, dx d\tau \right)^{1/2}, \end{aligned} \tag{2.90}$$

where, without loss of generality, we assumed that

$$\int_0^t \int_{\Omega_K} w_K^4\, dx d\tau \ge \int_0^t \int_{\Omega_K} w_L^4\, dx d\tau . \tag{2.91}$$

In (2.90) the first factor on the right hand side is bounded according to estimate (2.53). The second factor can be estimated as

$$\begin{aligned} \int_0^t \int_{\Omega_K} |w_K|^4\, dx d\tau &= \int_0^t \|w_K\|_{L^6}^4 (\text{mes } \Omega_K)^{\frac{1}{3}}\, d\tau \\ &\le \frac{1}{4} \int_0^t 2^{\frac{4}{3}} \|w_{Kx}\|^{\frac{4}{3}} \|w_K\|^{\frac{8}{3}} 2^{\frac{2}{3}} K^{-2} \|w_{Kx}\|^{\frac{2}{3}} \|w_K\|^{\frac{4}{3}}\, d\tau \\ &= K^{-2} \int_0^t \|w_{Kx}\|^2 \|w_K\|^4\, d\tau \xrightarrow{K \to \infty} 0, \end{aligned} \tag{2.92}$$

where we used Hölder's inequality with $f=1$, $p=3$ and $g=|w_K|^4$, $q=3/2$ in the first step and inequality (2.89) and (2.99) in the second step.

□

Lemma 2. *Under the hypotheses of Theorem 1*

$$\int_0^t \int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x\right) w_{xx}\, dx\, d\tau \xrightarrow{K,L\to\infty} 0 \tag{2.93}$$

for any $t>0$.

Proof of Lemma 2. We have, similarly as in Lemma 1,

$$\int_0^t \int_0^1 \left((f_K(w_K))_x - (f_L(w_L))_x\right) w_{xx}\, dx\, d\tau = \int_0^t \int_{\Omega_K\setminus\Omega_L} w_L w_{Lx} w_{xx}\, dx\, d\tau$$
$$\leq \left(\int_0^t \|w_{xx}(\tau)\|^2 d\tau\right)^{1/2} \left(\int_0^t \int_{\Omega_K\setminus\Omega_L} |w_L|^2 |w_{Lx}|^2\, dx\, d\tau\right)^{1/2} \tag{2.94}$$

The boundedness of the first factor above can be obtained integrating (2.32) which holds for all w_K uniformly in K. For the second factor we have

$$\int_0^t \int_{\Omega_K\setminus\Omega_L} |w_L|^2 |w_{Lx}|^2\, dx\, d\tau \quad\leq\quad \left(\int_0^t \int_{\Omega_K\setminus\Omega_L} |w_L|^4\, dx\, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|w_{Lx}\|_{L^4}^4\, d\tau\right)^{\frac{1}{2}}. \tag{2.95}$$

We already know from inequality (2.92) of Lemma 1 that the first factor on the right hand side of inequality (2.95) converges to zero as $K,L\to\infty$. The second factor is estimated with the help of inequality (2.98) as

$$\int_0^t \|w_{Lx}\|_{L^4}^4\, d\tau \leq M \int_0^t \left(\|w_x(\tau)\|^4 + \|w_{xx}(\tau)\|\, \|w_x(\tau)\|^3\right) d\tau$$
$$\leq Mt \max_{\tau\in[0,t]} \|w_x(\tau)\|^4 + M\left(\int_0^t \|w_{xx}(\tau)\|^2 d\tau\right)^{\frac{1}{2}} \sqrt{t} \max_{\tau\in[0,t]} \|w_x(\tau)\|^6, \tag{2.96}$$

where M is a generic constant. Since each expression is finite in (2.96) for any $t>0$, the result of Lemma 2 follows.

□

Lemma 3. *For any $w\in H^1(0,1)$ and $2\leq q\leq\infty$ we have*

$$\|w\|_{L^q} \leq \gamma_1 \|w\| + \gamma_2 \|w_x\|^{\alpha} \|w\|^{1-\alpha}, \tag{2.97}$$

where $\alpha = 1/2 - 1/q$, $\gamma_1 = 2^{1+\alpha}$ and $\gamma_2 = 2^{\frac{1}{2}} 6^{\frac{\alpha}{2}}$. We also have

$$\|w\|_{L^q}^2 \leq 2^{1+2\alpha} \|w\|^2 + 2^{1+\alpha} 3^{\alpha} \|w_x\|^{2\alpha} \|w\|^{2-2\alpha}. \tag{2.98}$$

Proof of Lemma 3. This is a 1–dimensional extension of a classical inequality (see, e.g., [13, Theorem 2.2, pp 62])

$$\|w\|_{L^q} \leq \beta \|w_x\|_{L^m}^{\alpha} \|w\|_{L^r}^{1-\alpha}, \tag{2.99}$$

which holds for $w \in W_m^1[a,b]$, $m \geq 1$ with $w(a) = 0$, where $r \leq q \leq \infty$,

$$\alpha = \left(\frac{1}{r} - \frac{1}{q}\right)\left(1 - \frac{1}{m} + \frac{1}{r}\right)^{-1} \tag{2.100}$$

and

$$\beta = \left(1 + \frac{m-1}{m} r\right)^{\alpha}. \tag{2.101}$$

As in [8], we consider an arbitrary $w \in H^1(0,1)$ and its extension

$$\widetilde{w}(x) = \begin{cases} w(x) & \text{if } x \in [0,1], \\ (x+1)\, w(-x) & \text{if } x \in [-1,0]. \end{cases} \tag{2.102}$$

Inequality (2.99) applies to $\widetilde{w}$ with $\alpha = 1/2 - 1/q$ and $r = 2 \leq q \leq \infty$, since $\widetilde{w} \in H^1[-1,1]$ and $\widetilde{w}(-1) = 0$. We have

$$\|\widetilde{w}\|_{L^q[-1,1]} \leq 2^{\alpha} \|\widetilde{w}_x\|_{L^2[-1,1]}^{\alpha} \|\widetilde{w}\|_{L^2[-1,1]}^{1-\alpha}. \tag{2.103}$$

We have the following relationship between the norms of $\widetilde{w}$ and w.

$$\|w\|_{L^q[0,1]} \leq \|\widetilde{w}\|_{L^q[-1,1]}, \tag{2.104}$$

$$\|\widetilde{w}\|_{L^2[-1,1]}^2 = \int_0^1 w^2(x)\, dx + \int_{-1}^0 |(x+1)\, w(-x)|^2 dx \leq 2\|w\|^2 \tag{2.105}$$

and

$$\begin{aligned} \|\widetilde{w}_x\|_{L^2[-1,1]}^2 &= \int_0^1 w_x^2(x)\, dx + \int_{-1}^0 |w(-x) - (x+1)\, w_x(-x)|^2 dx \\ &\leq \|w_x\|^2 + 2\int_{-1}^0 \left(\|w(-x)\|^2 + (x+1)^2 \|w_x(-x)\|^2\right) dx \\ &\leq \|w_x\|^2 + 2\left(\|w\|^2 + \|w_x\|^2\right) \\ &= 3\|w_x\|^2 + 2\|w\|^2. \end{aligned} \tag{2.106}$$

Combining inequalities (2.103)–(2.106) we obtain

$$\begin{aligned} \|w\|_{L^q[0,1]}^2 &\leq 2^{2\alpha}\left(3\|w_x\|^2 + 2\|w\|^2\right)^{\alpha} 2^{1-\alpha} \|w\|^{2-2\alpha} \\ &\leq 2^{1+2\alpha}\|w\|^2 + 2^{1+\alpha} 3^{\alpha} \|w_x\|^{2\alpha} \|w\|^{2-2\alpha} \end{aligned} \tag{2.107}$$

and

$$\begin{aligned} \|w\|_{L^q[0,1]} &\leq 2^{\alpha}\left(3\|w_x\|^2 + 2\|w\|^2\right)^{\frac{\alpha}{2}} 2^{\frac{1-\alpha}{2}} \|w\|^{1-\alpha} \\ &\leq 2^{\alpha}\left(3^{\frac{\alpha}{2}}\|w_x\|^{\alpha} + 2^{\frac{\alpha}{2}}\|w\|^{\alpha}\right) 2^{\frac{1-\alpha}{2}} \|w\|^{1-\alpha} \\ &= 2^{\alpha+1}\|w\| + 2^{\frac{1}{2}} 6^{\frac{\alpha}{2}} \|w_x\|^{\alpha} \|w\|^{1-\alpha}. \end{aligned} \tag{2.108}$$

□

3 A Numerical Example

In this section we compare three controllers through a numerical example: controller (2.3)–(2.5), controller (2.6)–(2.8) and controller (2.9)–(2.11). A comparison is also made relative to the uncontrolled system consisting of the KdVB equation (2.1) and the boundary conditions

$$\begin{aligned} w(0,t) &= 0, && (3.1)\\ w_x(1,t) &= w_0'(1)\,, && (3.2)\\ w_{xx}(1,t) &= w_0''(1)\,. && (3.3) \end{aligned}$$

The existence of a solution of the uncontrolled system is obvious, at least on a finite time interval. It can be proven for example using Galerkin's method. Note that boundary conditions (2.3)–(2.5) also stabilize the KdVB equation, as it was already reported in [15].

As a consequence of the third derivative in x and first derivative in t, it is necessary to use very small time steps ($\approx 10^{-9}$) in order to balance the very small number in the denominator resulting from the cube of the spatial step. We are able to compensate in a certain extent the very small time steps by rescaling the equation, i.e. compressing the time domain. We consider, from the above reason, the scaled equation

$$u_t - \varepsilon' u_{xx} + \delta' u_{xxx} + p u u_x = 0\,, \qquad x \in [0,1]\,, \quad t > 0\,, \tag{3.4}$$

with some initial data

$$u(x,0) = u_0(x)\,, \qquad u_0(0) = 0\,, \tag{3.5}$$

and in the controlled case with boundary condition

$$\begin{aligned} u(0,t) &= 0, && (3.6)\\ u_x(1,t) &= -\frac{p}{\varepsilon'}\left(c + \frac{1}{9c}u^2(1,t)\right)u(1,t)\,, && (3.7)\\ u_{xx}(1,t) &= \frac{p^2}{\varepsilon'^2}\left(c + \frac{1}{9c}u^2(1,t)\right)^2 u(1,t)\,, && (3.8) \end{aligned}$$

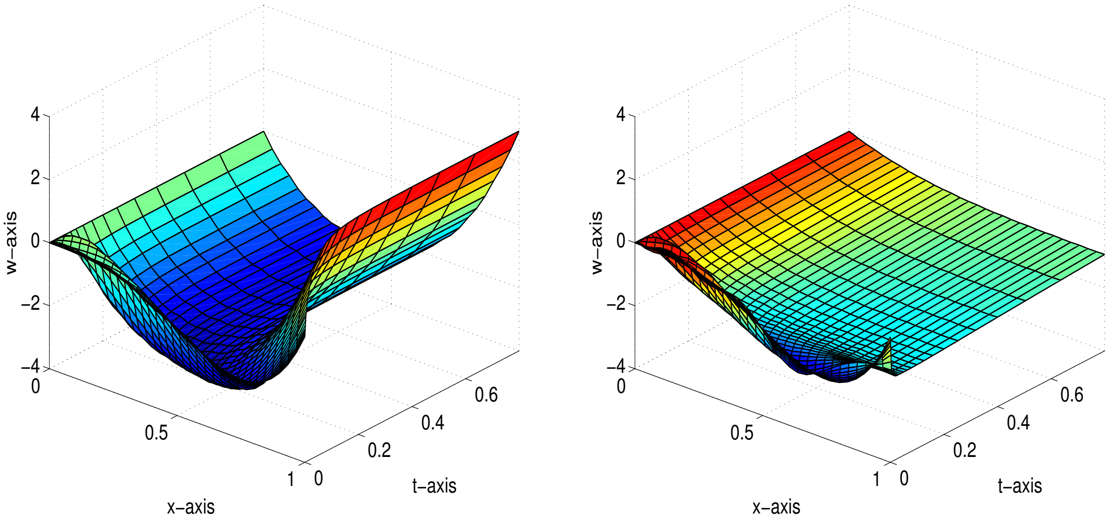
where ε', δ', c and p are positive constants. The transformation $u(x,t) \equiv w(x,pt)$ shows the equivalence of system (2.1), (2.2), (2.9), (2.10), (2.11) to (3.4)–(3.8) with $\varepsilon \equiv \varepsilon'/p$ and $\delta \equiv \delta'/p$.

Our numerical simulation is based on a fully discrete, implicit scheme of second order accuracy, using three time level quadratic approximation in time and central difference scheme in space, which is derived using the finite volume method (see, e.g. [11]). In accordance with the finite volume method we consider equation (3.4) integrated with respect to x over a small interval (which is called control volume, usually denoted by $[w,e]$ and whose center is a grid point). We obtain

$$\int_w^e u_t\,dx - \varepsilon u_x|_w^e + \delta' u_{xx}|_w^e + p\int_w^e u u_x\,dx = 0\,. \tag{3.9}$$

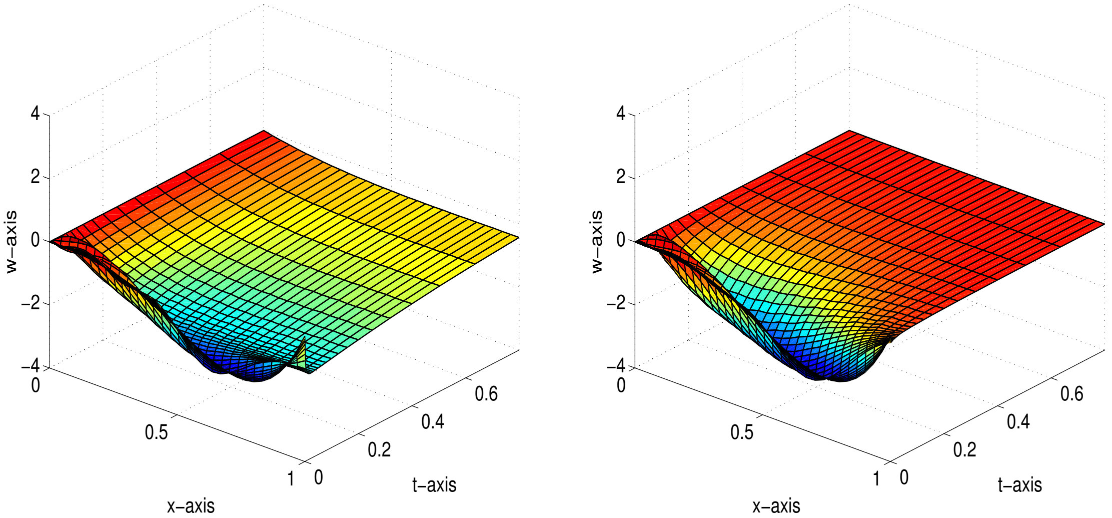
In an implicit scheme an important goal is to keep most of the expressions at the highest time level. Keeping this in mind, the last (quadratic) term of (3.9) is linearized in the following way:

$$\int_w^e u u_x\,dx = \frac{1}{2}u^2\Big|_w^e = \frac{1}{2}u^{n+1}u^n\Big|_w^e\,, \tag{3.10}$$



(a) Uncontrolled System

(b) Second Derivative Controlled (Quadratic)



(c) Second Derivative Controlled (Cubic)

(d) Two Derivatives Controlled

Figure 1: Comparison of Solutions

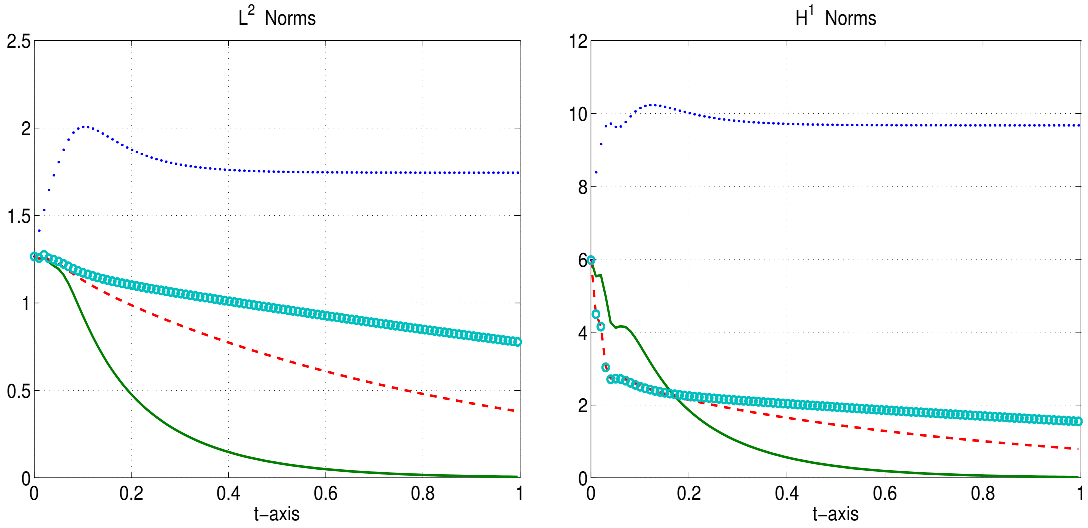


Figure 2: Comparison of Norms. ··· Uncontrolled, **ooo** Controlled Second Derivative (Quadratic), **- - -** Controlled Second Derivative (Cubic), **—**Two Derivatives Controlled

where the superscript $n+1$ denotes the highest time level. Using the notation $\vec{U}^n \equiv \left(u_1^n, u_2^n, \ldots, u_N^n\right)^T \equiv \left(u\left(t_n,x_1\right), u\left(t_n,x_2\right), \ldots, u\left(t_n,x_N\right)\right)^T$ and the aforementioned discretization and linearization, (3.9) gives us the following difference equation on a uniform grid:

$$
\begin{aligned}
&h\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{k} - \varepsilon' \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h} + \delta' \frac{u_{i+2}^{n+1} - 2u_{i+1}^{n+1} + 2u_{i-1}^{n+1} - u_{i-2}^{n+1}}{2h^2} \\
&+ \frac{p}{8}\left(u_{i+1}^{n+1} + u_i^{n+1}\right)\left(u_{i+1}^n + u_i^n\right) - \frac{p}{8}\left(u_i^{n+1} + u_{i-1}^{n+1}\right)\left(u_i^n + u_{i-1}^n\right) = 0 \qquad (3.11)
\end{aligned}
$$

where $n = 2,3,\ldots,i = 2,3,\ldots,N-2$. We use the notation k for the increment in time and h for the space mesh size. Equation (3.11) provides us a linear system with a sparse, band state matrix. Two initial vectors are required to start our three time level scheme: $\vec{U}^1$ and $\vec{U}^2$. The first vector $\vec{U}^1$ is the initial function $u_0(x)$ itself, and $\vec{U}^2$ was obtained from this vector using the one step Euler method.

At the boundaries $x = 0$ and $x = 1$ we use half control volumes. The boundary conditions at $x = 1$ provide a straightforward modification in the linear system after the linearization

$$
\begin{aligned}
u_x^{n+1}\big|_{x=1} &= g_x^n u_N^{n+1}, &(3.12)\\
u_{xx}^{n+1}\big|_{x=1} &= g_{xx}^n u_N^{n+1}, &(3.13)
\end{aligned}
$$

where

$$
\begin{aligned}
g_x^n &\equiv -\frac{p}{\varepsilon'}\left(c + \frac{1}{9c}\left(u_N^n\right)^2\right), &(3.14)\\
g_{xx}^n &\equiv \frac{p^2}{\varepsilon'^2}\left(c + \frac{1}{9c}\left(u_N^n\right)^2\right)^2. &(3.15)
\end{aligned}
$$

At the boundary $x = 0$, in order to obtain the necessary value outside the interval $[0,1]$, we use extrapolation based on the equation (3.4) reduced to the ODE

$$
-\varepsilon' u_{xx} + \delta' u_{xxx} = 0. \qquad (3.16)
$$

We omit the details of these simple calculations. The boundary conditions of the uncontrolled system and system (2.1)–(2.8) are handled similarly. The resulting sparse linear system is solved using the preconditioned BiConjugate Gradient Stabilized method implemented in a C++ templated library ([2, 10]). The computation was performed on a 300MHz, 130Mb memory Sun SPARC Workstation and, due to the small time step, it required several hours to reach one time unit with the numerical solution. Later these computations were repeated on a Silicon Graphics Workstation having two 250MHz MIPS R10000(IP30) CPUs and 384Mb main memory.

As an example, we consider the (KdVB) equation (3.4) with parameters $\varepsilon' = 1$, $\delta' = 10$, $p = 100$ and with initial function

$$u_0(x) = 20x^3(x - 1.001) . \tag{3.17}$$

The time step we use is $k = 10^{-9}$ with final time $T = 10^{-2}$, and spatial step $h = 5 \times 10^{-3}$. The scaling $p = 100$ corresponds to an unscaled Korteweg–de Vries–Burgers system with parameters $\varepsilon = 0.01$, $\delta = 0.1$ on a time interval $[0, 1]$. In the controlled case the control gain was $c = 0.1$. As we can see in Figure 1, the uncontrolled solution seems to converge to a nontrivial stationary solution. While all three controlled systems converge to zero (parts (b), (c) and (d) of Figure 1), cases (b) and (c), when the first derivate is kept at zero at $x = 1$ and only the second derivative is controlled by feedback, show poor convergence relative to our controller (3.7)–(3.8). In fact, Figure 2 shows that the differences between the rates of convergence are significant both in the L^2 and in the H^1 sense.

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