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Nonlinear Analysis: Real World Applications 7 (2006) 1005 – 1028

Nonlinear Analysis
Real World Applications

www.elsevier.com/locate/na

Forced nonlinear oscillations of elastic membranes☆

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Received 1 February 2005; accepted 14 September 2005

Abstract

An initial-boundary-value problem is considered for the spatially two-dimensional damped Boussinesq equation with a forcing term. The problem in question is argued to serve as a model for describing small nonlinear oscillations of a circular membrane under the influence of acoustic pressure. Eigenfunction expansion method is used for constructing the global-in-time solution of the problem in question. Existence and uniqueness follow from the construction. Long-time asymptotics is computed on the basis of the obtained series representation. Numerical simulations are conducted. The algorithm shows excellent convergence properties.

Keywords: Circular elastic; Membrane; Nonlinear oscillations; Forced Boussinesq equation; Global solutions

1. Introduction

The present work is concerned with the problem of response of an elastic membrane to an incident acoustic wave. This issue appears in the context of studying acoustic detection via registering vibrations of thin elastic structures [10,24]. In particular, the necessity of creation of passive sensor technologies determines the importance of the study of oscillations of elastic membranes under the influence of acoustic fields. Unattended acoustic ground sensors can have as their basic element a membrane with fixed ends (clamped, simply supported or even nonlinearly damped). Circular geometry considered in the present paper is not the only option. Indeed, geometric optimization (shape or form finding) can be another branch of research in this area [10]. However, the present study is restricted to response analysis for a fixed geometric configuration, namely, for a circular membrane with a simply supported boundary.

The so called "good" Boussinesq equation is well known in the context of governing small nonlinear oscillations of elastic beams. This equation can be written in the form (see [8,35])

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.1}$$

where $\alpha = const > 0$ is the dispersion parameter depending on the compression and rigidity characteristics of the material, $\beta = const \in \mathbf{R}$ is the constant coefficient controlling nonlinearity (it can be set equal to one by appropriate scaling, but it is convenient to keep it in order to trace the influence of nonlinearity) and $u(x, t)$ is the vertical deflection. The quadratic nonlinearity appearing in (1.1) accounts for the curvature of the bending beam. Eq. (1.1) with $\alpha < 0$ is

☆ This work is supported by the Army Research Office under Research Grant No. DAAD 19-03-1-0204.
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doi:10.1016/j.nonrwa.2005.09.006

known as the "bad" Boussinesq equation since it possesses linear instability. It describes the propagation of long surface waves on shallow water.

The one-dimensional in space Boussinesq equation and its generalization

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + (f(u))_{xx} \tag{1.2}$$

have been the subject of many papers (see [4,7,9,17,19,20,22,23,27] and the references therein). An overview of these results can be found in [31].

A two-dimensional "bad" Boussinesq equation

$$u_{tt} = u_{xxxx} + u_{xx} + u_{yy} + 3(u^2)_{xx} \tag{1.3}$$

was introduced in [11] to describe the propagation of gravity waves on the surface of water, in particular the head-on collision of oblique waves. In this model the main wave propagation takes place in the x-direction with weak transverse effects in the y-direction (like the Kadomtsev–Petviashvili equation KPII, see [13]).

More recently, there appeared some results on the forced Boussinesq equation in one and two space dimensions. An initial-value problem for the "bad" Boussinesq equation with a potential,

$$u_{tt} = 3\Delta^2 u + \Delta u - 12(V(x) * \Delta(|u|^{\lambda} u)), \quad x \in \mathbf{R}^2, \quad t > 0,$$

was considered in the paper [1]. For small initial data, $\lambda > 8$ and sufficiently smooth potential $V(x)$ it was shown that there exists a global solution satisfying the decay estimate $\|u(\cdot, t)\|_{L_\infty} \leqslant c(1+t)^{-7}$.

A generalized forced "good" Boussinesq equation in one space dimension

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + (f(u))_{xx} + h(x, t) \tag{1.4}$$

with $\alpha = 1$ was considered in [12]. Existence of periodic traveling wave solutions of the form $u = \phi(\xi)$, where $\xi = x - ct$, was established via analyzing the corresponding integral equation and applying the fixed point theorem.

In many practical situations damping effects are comparable in strength to nonlinear and dispersive ones and in such cases (1.1) requires a dissipative term [3]. A convenient item responsible for internal friction and related to heat generation (see, e.g., [8]), is $-2bu_{txx}$, where $b = const > 0$. After adding it to the left-hand side of (1.1) the equation becomes

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}. \tag{1.5}$$

Pego and Weinstein [23] studied the behavior of solitary waves described by (1.5) with a more general nonlinearity $(f(u))_{xx}$. As regards viscosity, another possibility may be to replace the second term in the left-hand side of (1.5) by $+2bu_{xxxxt}$ (see [8]) corresponding to stronger friction. We will restrict ourselves to weaker damping keeping in mind that the case of stronger friction is sufficiently easier for the current analysis.

A two-dimensional analog of (1.5) (sometimes called a damped plate equation [14,15]) is

$$u_{tt} - 2b\Delta u_t = -\alpha\Delta^2 u + \Delta u + \beta\Delta(u^2), \tag{1.6}$$

where $\alpha, b = const > 0$, $\Delta = \partial_x^2 + \partial_y^2$, $\Delta^2 = \Delta\Delta$ and $u(x, y, t)$ is the vertical deflection of a membrane. It appears in the context of modeling small nonlinear oscillations of elastic membranes. A "uniform" nature of this equation contrasts with that of its "asymmetric" counterpart (1.3) from the theory of water waves. An initial-boundary-value problem for (1.6) in a unit disc Ω with homogeneous boundary conditions and small initial data was considered in [31], and the global-in-time mild solutions were constructed in the Sobolev space $H^s(\Omega)$ with $s < 1$. A general description of the method applied and an overview of various problems that can be dealt with can be found in the paper [30]. In particular, the papers [28,31,30] dealt with the homogeneous equation (1.6) in bounded domains in one and several space dimensions, while in [29] this approach was used for solving a nonlocal equation, namely a fractional Laplacian semi-linear heat equation. The first part of the method resembles the classical method of separation of variables. However, the necessity to solve nonlinear integral equations brings new ideas into consideration and emphasizes the difference from Galerkin's method (see [22] for comparison). Indeed, the latter one is based on projection onto the finite-dimensional space of eigenvectors and allows one to prove existence, uniqueness and regularity of solutions. Our method utilizes projection onto an infinite-dimensional space of eigenvectors with the subsequent detailed analysis of decay of the coefficients of the corresponding series. These coefficients are defined via integral equations. The main emphasis falls on construction of solutions while existence and uniqueness simply follow from it. Another point of interest is that the long-time asymptotic expansion can be obtained from the representation in question.

One of the main issues addressed in the present investigation is the question of regularity. In the earlier paper [31] rather complicated requirements were imposed on the initial data (no source term was present in the equation). They involved total bounded variation with respect to the radial coordinate r and absolute integrability in the angle θ. All this secured convergence of the eigenfunction series representing the solution in $H^s(\Omega)$ with $s<1$. Below we shall assume only that the source term belongs to $L_2(\Omega)$ with respect to (r,θ) and prove that the constructed mild solutions belong to $H^s(\Omega)$, $s<\frac{3}{2}$. To this end we shall employ a new special function introduced in [32], namely, a convolution of Rayleigh functions with respect to the Bessel index.

For a given Bessel function $J_m(x)$ Rayleigh functions are defined by the formula (see, e.g., [33, p. 502]; and the references in [32])

$$\sigma_l(m)=\sum_{n=1}^{\infty}\frac{1}{\lambda_{mn}^{2l}},$$

where λ_{mn} is the zero of the Bessel function and n is the number of zero. Rayleigh functions have been used in studying classical problems of vibrating drumheads, heat conduction in cylinders, normal modes in resonant cavities and Fraunhofer diffraction through circular apertures. The convolution in question can be written as

$$R(m)=\sum_{k=-\infty}^{\infty}\sigma_1(m-k)\sigma_1(k)=\sum_{\substack{p,k\in\mathbf{Z};q,s\geqslant 1;\\ p+k=m}}\frac{1}{\lambda_{pq}^2}\frac{1}{\lambda_{ks}^2}, \tag{1.7}$$

where $p,k,m\in\mathbf{Z}$ are the indices of angular eigenfunctions. Function (1.7) appears as a result of treating quadratic nonlinearities in semi-linear evolution equations in circular domains and using the series of eigenfunctions of the Laplace operator in a disc. This special function is responsible for the smoothness transfer due to periodicity in the angular coordinate θ. Its convolution structure is a consequence of the orthogonality of the angular eigenfunctions $\{e^{im\theta}\}_{m=-\infty}^{\infty}$. However, we must point out that the combined influence of the nonlinearity and the circular geometry does not allow to reach strong solutions.

In contrast to the situation with the nonhomogeneous Korteweg–de Vries equation (see [5,6] and the references therein), the theory of the forced Boussinesq equation is much less developed. In the present work the approach proposed in [28–31] is extended to investigation of the nonhomogeneous equation (1.4) in two space dimensions and with $f(u)=u^2$. Studying nonhomogeneous initial conditions would not be a problem but would make our formulas somewhat unwieldy (it can be done along the lines of [31] with improvements offered in the current work). For a special choice of the source term $f(r,\theta,t)=F(r,\theta)\cos(\omega t)e^{-\kappa t}$ with small $\kappa>0$, the long-time asymptotics of the solution are computed. On one hand, this forcing corresponds to a rather typical situation when a membrane is excited by an incident acoustic field slowly decaying in time due to absorption. On the other hand, it provides a convenient example of separation of variables and makes the interpretation easier.

In the current paper we test the method in question numerically for the first time. Simulations are performed for several forcing terms and the effectiveness of the proposed algorithm is studied. An important feature of our approach consists in providing an explicit convergence rate for the nonlinear problem.

The paper is organized as follows. In Section 2 the main initial-boundary-value problem is posed, notations are introduced and the function spaces are defined. Some properties of eigenfunctions of the Laplace operator in a disc are presented. Section 3 is devoted to the proof of global-in-time existence and uniqueness for the problem in question and to the construction of solutions (Theorem 1). Auxiliary statements concerning representation and estimates of the eigenfunction expansion coefficients are collected in Lemmas 1–5 and are given in the same section. Lemma 4 provides a closed form representation of the special function $R(m)$. Lemma 5 with $\kappa=0$ is used for proving convergence of the eigenfunction series. The same lemma with $\kappa>0$ is used for computing the asymptotics. Calculating the long-time asymptotics for the special choice of the source term forms the contents of Section 4 (Theorem 3). Results of numerical simulations are presented in Section 5.

2. Problem statement and preliminaries

Denote by Ω a disk of a unit radius and put the origin of the coordinate system in its center, so that in polar coordinates $\Omega = \{(r, \theta) : r < 1, \theta \in [-\pi, \pi]\}$. Let $\partial\Omega$ denote its boundary, i.e. $\partial\Omega = \{(r, \theta) : r = 1, \theta \in [-\pi, \pi]\}$. Our purpose is to consider the following initial-boundary-value problem for the real function u:

$$\begin{aligned}
&u_{tt} - 2b\Delta u_t = -\alpha \Delta^2 u + \Delta u + \beta \Delta(u^2) + af, \quad (r, \theta) \in \Omega, \quad t > 0,\\
&u(r, \theta, 0) = u_t(r, \theta, 0) = 0, \quad (r, \theta) \in \Omega,\\
&u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \quad t > 0,\\
&u(r, \theta + 2\pi, t) = u(r, \theta, t), \quad (r, \theta) \in \Omega, \quad t > 0,\\
&|u(0, \theta, t)| < \infty,
\end{aligned} \tag{2.1}$$

where $\alpha, b, a = const > 0$, $\beta = const \in \mathbf{R}$ and $f = f(r, \theta, t)$ is a real function. Parameter a controls the forcing term and must be bounded in order to guarantee convergence of a certain series. It constitutes a natural assumption since large forcing terms can cause a blow up. In other words, acoustic pressure is not supposed to be large in the framework of the current model. The above boundary conditions correspond to a simply supported boundary (see [2,16]).

We restrict our attention to considering the relation $\alpha > b^2$ which corresponds to small damping. It is the most interesting case from both mathematical and physical points of view and corresponds to the existence of an infinite number of damped oscillations. If the inverse relation holds (the so called overdamping case), aperiodic processes play the main role.

According to the method proposed in [29–31], we seek solutions of the problem (2.1) in the form of a series

$$u(r, \theta, t) = \sum_{m,n} \hat{u}_{mn}(t)\chi_{mn}(r, \theta). \tag{2.2}$$

Here and in the sequel the notation $\sum_{m,n}$ is used for the double sum $\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}$ and $\chi_{mn}(r, \theta)$ are the eigenfunctions of the Laplace operator in a disc, namely nontrivial solutions of the problem

$$\begin{aligned}
&\Delta\chi = -\Lambda\chi, \quad (r, \theta) \in \Omega,\\
&\chi|_{\partial\Omega} = 0,\\
&\chi(r, \theta + 2\pi) = \chi(r, \theta), \quad (r, \theta) \in \Omega,\\
&|\chi(0, \theta)| < \infty
\end{aligned} \tag{2.3}$$

corresponding to the eigenvalues

$$\Lambda_{mn} = \lambda_{mn}^2.$$

They are

$$\chi_{mn}(r, \theta) = J_m(\lambda_{mn} r)e^{im\theta}, \quad m \in \mathbf{Z}, \quad n \in \mathbf{N}, \tag{2.4}$$

where $J_m(x)$ are Bessel functions of index m and λ_{mn} are their positive zeros numbered in the order of increasing magnitudes ($n = 1, 2, \ldots,$ is the number of the zero). The appearance of double indices reflects the influence of the angular eigenfunctions on their radial counterparts. In this way we satisfy the boundary and periodicity conditions in (2.1). Note that the following property holds for $m \in \mathbf{N}$:

$$J_{-m}(z) = (-1)^m J_m(z). \tag{2.5}$$

Introduce the real space $L_2(0, 1)$ equipped with the inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$. We also need the complex space $L_2(\Omega)$ endowed with the inner product $\langle\cdot, \cdot\rangle$ and the norm $\|\cdot\|_0$. Then, according to [26], the following relations hold:

$$\|\chi_{mn}\|_0^2 = 2\pi\|J_m(\lambda_{mn}\cdot)\|^2,$$

$$\|J_m(\lambda_{mn}\cdot)\|^2 = \int_0^1 r J_m^2(\lambda_{mn} r)\,\mathrm{d}r = \frac{1}{2}J_{m+1}^2(\lambda_{mn}).$$

For sufficiently large $q > 0$ there exist such positive constants $\mathscr{C}_1$ and $\mathscr{C}_2$ that (see [26])

$$\frac{\mathscr{C}_1}{q} \leqslant \| J_m(q\cdot)\|^2 \leqslant \frac{\mathscr{C}_2}{q}. \tag{2.6}$$

In the sequel we shall need asymptotic expansions of the zeros of the Bessel functions $J_m(z)$, $m \geqslant 0$. For $m \ll n$ large positive zeros of $J_m(z)$ have the following asymptotic expansion uniform in m (McMahon's expansion, see [21, p. 247]):

$$\lambda_{mn} = \upsilon_{mn} + O\left(\frac{1}{\upsilon_{mn}}\right),$$

where

$$\upsilon_{mn} = \left(m + 2n - \frac{1}{2}\right)\frac{\pi}{2} \quad \text{for } n \to +\infty. \tag{2.7}$$

Since representation of the special function (1.7) involves the ψ-function, we would like to recall the definition of the latter and a couple of its properties. The function $\psi(z)$ is defined as a logarithmic derivative of the Γ-function, namely for $z \in \mathbf{C}$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

It is analytic in the whole complex plane excluding the points $z = 0, -1, -2, \ldots,$ where it has simple poles. For integers $m \geqslant 1$ the following formula holds:

$$\psi(m+1) + \gamma = \sum_{k=1}^{m} \frac{1}{k},$$

where $\gamma = 0.577215\ldots$ is the Euler constant. For $m = 0$ the relation is $\psi(1) + \gamma = 0$.

The derivative of $\psi(z)$ is defined by the series

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}.$$

Therefore, it is easy to see that

$$\psi'(1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \frac{\pi^2}{6}. \tag{2.8}$$

We shall also use Sobolev spaces $W_2^s(\Omega) = H^s(\Omega)$, $s \in \mathbf{R}$, equipped with the norm (see [18])

$$\|\mathscr{F}\|^2_{H^s(\Omega)} = \|\mathscr{F}\|^2_s = \sum_{m,n} \lambda_{mn}^{2s} |\widehat{\mathscr{F}}_{mn}|^2 \|\chi_{mn}\|_0^2,$$

where $\widehat{\mathscr{F}}_{mn}$ are the complex Fourier–Bessel coefficients of the function $\mathscr{F}$, namely

$$\hat{\mathscr{F}}_{mn} = \frac{\langle \mathscr{F}, \chi_{mn}\rangle}{\|\chi_{mn}\|_0^2}.$$

Note that [33] $\lambda_{mn} > 0$ for all $m \in \mathbf{Z}$, $n \in \mathbf{N}$.

Let $H_0^s(\Omega)$, $s \geqslant 0$, be the completion of the space $C_0^\infty(\Omega)$ endowed with the inner product $\langle \cdot, \cdot \rangle_s$ and the corresponding norm $\|\cdot\|_s$. For the case $s = 0$ the subindex will be omitted in the notation of the inner product, i.e. we shall write simply $\langle \cdot, \cdot \rangle$. By Lax's theorem [34], the dual space $H_0^s(\Omega)' = H_0^{-s}(\Omega)$, $s \geqslant 0$, of the space $H_0^s(\Omega)$ can be identified with the completion of the space $H_0^0(\Omega)$ with respect to the negative norm $\|\cdot\|_{-s}$. We shall also need the space $C_b(\mathbf{R}^+, H^s(\Omega))$ equipped with the norm

$$\|u\|_{C_b(\mathbf{R}^+, H^s(\Omega))} = \sup_{t \geqslant 0} \|u(\cdot, t)\|_s.$$

3. Main results

Integration of Eq. (2.1) with respect to t leads to a nonlinear integral equation which serves for the definition of a mild solution of problem (2.1). Denote by A the operator $-\Delta$ defined on sufficiently smooth functions $\chi(r, \theta)$ satisfying the Dirichlet and periodicity conditions (2.3) and let

$$\sigma(A) = (\mu A^2 + A)^{1/2} \quad \text{where } \mu = \alpha - b^2 > 0.$$

Definition. A function $u(t)$ is called a *mild solution* of the problem (2.1) if it satisfies the integral equation

$$\begin{aligned} u(t) = a \int_0^t & \exp[-b(t-\tau)A](\sigma(A))^{-1} \sin(\sigma(A)(t-\tau)) f(\tau)\, \mathrm{d}\tau \\ & - \beta \int_0^t \exp[-b(t-\tau)A](\sigma(A))^{-1} \sin(\sigma(A)(t-\tau)) A u^2(\tau)\, \mathrm{d}\tau \end{aligned}$$

in the Banach space $C_b(\mathbf{R}^+, H_0^s(\Omega))$.

Our main result is the following theorem.

Theorem 1. *If* $\alpha > b^2$, $f \in C_b(\mathbf{R}^+, L_2(\Omega))$, *and* $a < (c_0 C_*)^{-1}$, *where* c_0 *and* C_* *are the constants defined by* (3.12), *then there exists a mild solution* $u \in C_b(\mathbf{R}^+, H_0^s(\Omega))$ *with* $s < \frac{3}{2}$ *of the problem* (2.1). *For* $-1 < s < \frac{3}{2}$ *this solution is unique. It can be represented in the form of a series*

$$u(r, \theta, t) = \sum_{m,n} \hat{u}_{mn}(t) J_m(\lambda_{mn} r) \mathrm{e}^{\mathrm{i}m\theta}, \tag{3.1}$$

where the coefficients $\hat{u}_{mn}(t)$ *are defined by* (3.17), (3.7) *and* (3.8) *and convergence is understood in the sense of* $H^s(\Omega)$.

The proof of the theorem is approached through a series of lemmas.

Lemma 1. *If* $\mathscr{F} \in L_2(\Omega)$, *then*

$$|\hat{\mathscr{F}}_{mn}| \leqslant \frac{\|\mathscr{F}\|_0 \sqrt{\lambda_{mn}}}{\sqrt{2\pi \mathscr{C}_1}}, \tag{3.2}$$

where $\mathscr{C}_1$ *is the constant from the estimate* (2.6).

Proof. Fourier–Bessel coefficients of the function $\mathscr{F}$ are defined as

$$\hat{\mathscr{F}}_{mn} = \frac{1}{\|\chi_{mn}\|_0^2} \int_\Omega \mathscr{F}(r, \theta) J_m(\lambda_{mn} r) \mathrm{e}^{-\mathrm{i}m\theta}\, \mathrm{d}\Omega, \quad m \geqslant 0, \quad n \geqslant 1. \tag{3.3}$$

Since $\mathscr{F} \in L_2(\Omega)$, by the Cauchy–Schwartz inequality,

$$\left| \int_\Omega \mathscr{F}(r, \theta) J_m(\lambda_{mn} r) \mathrm{e}^{-\mathrm{i}m\theta}\, \mathrm{d}\Omega \right| \leqslant \left(\int_\Omega |\mathscr{F}|^2\, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega |J_m(\lambda_{mn} r)|^2\, \mathrm{d}\Omega \right)^{1/2} \leqslant \frac{C}{\sqrt{\lambda_{mn}}} \|\mathscr{F}\|_0. \tag{3.4}$$

It is now easy to find the bound of (3.3) with the help of (3.4) and (2.6). Thus (3.2) is established. □

The analysis to follow is essentially based on the expansion of the nonlinearity u^2 into the eigenfunction series. This involves the use of the coefficients

$$b(m, n; p, q, k, s) = \frac{\langle \chi_{pq} \cdot \chi_{ks}, \chi_{mn} \rangle}{\|\chi_{mn}\|_0^2} = \frac{\int_0^1 J_m(\lambda_{mn} r) J_p(\lambda_{pq} r) J_l(\lambda_{ls} r) r\, \mathrm{d}r}{\|J_m(\lambda_{mn} \cdot)\|^2}. \tag{3.5}$$

Lemma 2. *The following inequality holds for all $m, p, k \in \mathbf{Z}$ and $n, k, s \in \mathbf{N}$*

$$|b(m,n;\, p,q,k,s)| \leqslant C\sqrt{\frac{\lambda_{mn}}{\lambda_{pq}\lambda_{ks}}},$$

where the constant C is independent of m, n, p, q, k and s.

Proof. See [31]. □

The next statement provides subtle estimates of the aforementioned coefficients and will be needed for proving uniqueness of solutions.

Lemma 3. *For any fixed integers $n \geqslant 1$, any integers $m, p, k \geqslant 0$; $q, s \geqslant 1$ and $\lambda_{pq}, \lambda_{ks} \to \infty$ there exists a constant C independent of m, n, p, q, k, s such that the following inequalities hold*:

$$|b(m,n;\, p,q,k,s)| \leqslant C\sqrt{\lambda_{mn}} \begin{cases} \lambda_{pq}^{-3/2}\lambda_{ks}^{-1/2}, & \lambda_{pq} > \lambda_{ks}, \\ \lambda_{pq}^{-1/2}\lambda_{ks}^{-3/2}, & \lambda_{pq} < \lambda_{ks}, \\ \lambda_{pq}^{-1}, & \lambda_{pq} = \lambda_{ks}. \end{cases}$$

Proof. Since the Bessel functions involved possess property (2.5), consideration of only nonnegative indices m, p, k does not represent a loss of generality. Therefore the proof is a slight modification of that of Lemma 4.4 of [30] which dealt with the estimate of the integral $\int_0^1 J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)J_k(\lambda_{ks}r)\sqrt{r}\,\mathrm{d}r$. □

Lemma 4. *The function $R(m)$ has the following representation for $m \in \mathbf{Z} \cup \{0\}$*:

$$R(0) = \frac{1}{16}\left(\frac{\pi^2}{3} - 1\right),$$

$$R(m) = \frac{1}{8(|m|+2)}\left[2(|m|+1)\,\frac{\gamma + \psi(|m|+1)}{|m|} - 1\right] \quad \textit{for } m \in \mathbf{Z}.$$

Proof. See [32]. □

Corollary 2. *The following asymptotics holds for $|m| \to \infty$*:

$$R(m) \sim \frac{\ln\,|m|}{4|m|} + \frac{2\gamma - 1}{8|m|} - \frac{\ln\,|m|}{4|m|^2} + O\left(\frac{1}{|m|^2}\right). \tag{3.6}$$

Remark 1. Consider an extension of $R(m)$ to the set $\mathbf{R}\backslash\{0\}$, i.e., the function $R(x)$ defined by the formula

$$R(x) = \frac{1}{8(|x|+2)}\left[2(|x|+1)\frac{\gamma + \psi(|x|+1)}{|x|} - 1\right].$$

Application of l'Hospital's rule and the formula (2.8) allows one to conclude that this function has a removable singularity at the point $x = 0$. Therefore it can be identified with a continuous function on $\mathbf{R}$. Also, $R(x) \leqslant R(0)$ for $x \in \mathbf{R}$, where $R(0) \simeq 0.1429$ (see [32]).

Set

$$\hat{v}_{mn}^{(0)}(t) = \int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)}\,\frac{\sin[\sigma_{mn}(t-\tau)]}{\sigma_{mn}}\,\hat{f}_{mn}(\tau)\,\mathrm{d}\tau, \tag{3.7}$$

$$\hat{v}_{mn}^{(N)}(t) = -\frac{\beta\lambda_{mn}^2}{\sigma_{mn}}\int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)}\,\sin[\sigma_{mn}(t-\tau)]\,Q_{mn}^{(N)}(\hat{v}(\tau))\,\mathrm{d}\tau \quad \text{for } N \geqslant 1, \tag{3.8}$$

where

$$\sigma_{mn} = \lambda_{mn}\sqrt{\mu\lambda_{mn}^2 + 1},$$

$$Q_{mn}^{(N)}(\hat{v}(t)) = \sum_{p,q,k,s}{}^{*} b(m,n;\,p,q,k,s)\sum_{j=1}^{N}\hat{v}_{pq}^{(j-1)}(t)\hat{v}_{ks}^{(N-j)}(t)$$

and the following notation has been used

$$\sum_{p,q,k,s}{}^{*} = \sum_{\substack{p,k\in\mathbf{Z};q,s\geqslant 1;\\ p+k=m}} . \tag{3.9}$$

Lemma 5. *Assume that for* $m\in\mathbf{Z}$, $n\geqslant 1$

$$|\hat{f}_{mn}(t)| \leqslant f_*\sqrt{\lambda_{mn}}\mathrm{e}^{-\kappa t},$$

where f_*, $\kappa = const$, *and* $0\leqslant 2\kappa < b\lambda_{01}^2$. *Then the functions* $\hat{v}_{mn}^{(N)}(t)$ *with* $m\in\mathbf{Z}$, $n\geqslant 1$, $t>0$ *satisfy the inequalities*

$$|\hat{v}_{mn}^{(0)}(t)| \leqslant c_0\frac{\mathrm{e}^{-\kappa t}}{\lambda_{mn}^{7/2}}, \tag{3.10}$$

$$|\hat{v}_{mn}^{(N)}(t)| \leqslant \frac{C_*^N c_0^{N+1}}{(N+1)^2}\frac{\mathrm{e}^{-2\kappa t}}{\lambda_{mn}^{3/2}}R(m) \quad \textit{for } N\geqslant 1, \tag{3.11}$$

where

$$c_0 = \frac{f_*}{\sqrt{\mu}b[1-\kappa/(b\lambda_{01}^2)]}, \qquad C_* = \frac{2\pi^2|\beta|}{3\sqrt{\mu}b[1-2\kappa/(b\lambda_{01}^2)]} \tag{3.12}$$

and the function $R(m)$ *is defined by* (1.7).

Proof. Substituting the estimate $|\hat{\mathscr{F}}_{mn}(t)| \leqslant \mathscr{F}_*\sqrt{\lambda_{mn}}\mathrm{e}^{-\kappa t}$ into (3.7) yields for $N=0$

$$\begin{aligned}
|\hat{v}_{mn}^{(0)}(t)| &\leqslant \mathscr{F}_*\sqrt{\lambda_{mn}}\mathrm{e}^{-b\lambda_{mn}^2 t}\int_0^t \mathrm{e}^{(b\lambda_{mn}^2-\kappa)\tau}\left|\frac{\sin(\sigma_{mn}(t-\tau))}{\sigma_{mn}}\right|\mathrm{d}\tau\\
&\leqslant \frac{\mathscr{F}_*(\mathrm{e}^{-\kappa t}-\mathrm{e}^{-b\lambda_{mn}^2 t})}{\lambda_{mn}^{7/2}\sqrt{\mu}[1-\kappa/(b\lambda_{01}^2)]} \leqslant \frac{c_0\mathrm{e}^{-\kappa t}}{\lambda_{mn}^{7/2}}.
\end{aligned}$$

Here we have used the fact that $\kappa/b\lambda_{mn}^2 \geqslant \kappa/b\lambda_{01}^2 > 0$, according to the hypothesis.

Next, we use the induction on the number *N*. For $N=1$ applying Lemma 1 we can write the following chain of inequalities

$$\begin{aligned}
|\hat{v}_{mn}^{(1)}(t)| &\leqslant \frac{\beta\lambda_{mn}^2}{\sigma_{mn}}\int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)}\sum_{p,q,k,s}{}^{*}|b(m,n;\,p,q,k,s)||\hat{v}_{pq}^{(0)}(\tau)||\hat{v}_{ks}^{(0)}(\tau)|\,\mathrm{d}\tau\\
&\leqslant \frac{|\beta|\lambda_{mn}\sqrt{\lambda_{mn}}}{b^2\sqrt{\mu\lambda_{mn}^2+1}}\mathbf{I}_{mn}(t)c_0^2\sum_{p,q,k,s}{}^{*}\frac{1}{\lambda_{pq}^2\lambda_{ks}^2}\\
&\leqslant c_0^2 C_* R(m)\frac{|\beta|}{\sqrt{\mu}b[1-2\kappa/(b\lambda_{01}^2)]}\frac{\mathrm{e}^{-2\kappa t}}{\lambda_{mn}^{3/2}}\\
&\leqslant c_0^2 C_* R(m)\frac{\mathrm{e}^{-2\kappa t}}{\lambda_{mn}^{3/2}}.
\end{aligned}$$

Here we have used the estimate

$$\mathbf{I}_{mn}(t)=\mathrm{e}^{-b\lambda_{mn}^2 t}\int_0^t \mathrm{e}^{(b\lambda_{mn}^2-2\kappa)\tau}\,\mathrm{d}\tau \leqslant \frac{\mathrm{e}^{-2\kappa t}-\mathrm{e}^{-b\lambda_{mn}^2 t}}{b\lambda_{mn}^2-2\kappa} \leqslant \frac{\mathrm{e}^{-2\kappa t}}{b\lambda_{mn}^2[1-2\kappa/(b\lambda_{01}^2)]}.$$

Assume now that the inequalities (3.11) hold for all $\hat{v}_{mn}^{(l)}(t)$ with $1\leqslant l\leqslant N-1$ and prove that they are true for $l=N$. Note that for $1\leqslant j\leqslant N$

$$\frac{1}{j^2(N+1-j)^2}\leqslant \frac{2}{(N+1)^2}\left[\frac{1}{j^2}+\frac{1}{(N+1-j)^2}\right].$$

Consequently, we can get an upper bound uniform in N

$$\sum_{j=1}^{N}\frac{1}{j^2(N+1-j)^2}\leqslant \frac{2}{(N+1)^2}\sum_{j=1}^{N}\left[\frac{1}{j^2}+\frac{1}{(N+1-j)^2}\right]\leqslant \frac{4}{(N+1)^2}\sum_{1}^{\infty}\frac{1}{j^2}=\frac{2\pi^2}{3(N+1)^2}.$$

Therefore,

$$\begin{aligned}|\hat{v}_{mn}^{(N)}(t)| &\leqslant \frac{|\beta|\lambda_{mn}\sqrt{\lambda_{mn}}}{\sqrt{\mu\lambda_{mn}^2+1}}\,\mathbf{I}_{mn}(t)\,\frac{2}{(N+1)^2}\sum_{j=1}^{N}C_*^{j-1}C_*^{N-j}c_0^{j}c_0^{N+1-j}\left[\frac{1}{j^2}+\frac{1}{(N+1-j)^2}\right]\sum_{p,q,k,s}{}^{*}\frac{1}{\lambda_{pq}^2\lambda_{ks}^2}\\ &\leqslant \frac{C_*^N c_0^{N+1}}{(N+1)^2}R(m)\mathrm{e}^{-2\kappa t}\leqslant \frac{C_*^N c_0^{N+1}}{(N+1)^2}R(m)\mathrm{e}^{-2\kappa t}.\end{aligned}$$

The proof is complete. □

Remark 2. We would like to emphasize the fact that the constants C_* and c_0 are independent of the iteration number N which is essential for the proof to follow.

Now we can prove Theorem 1.

Proof of Theorem 1. (i) *Existence and construction of solutions*: Our main idea consists in expanding the nonlinearity u^2 into the eigenfunction series and deducing the estimates of its coefficients. We can write that

$$u^2=\sum_{m,n}\widehat{u^2}_{mn}(t)\chi_{mn}(r,\theta),$$

where

$$\begin{aligned}\widehat{u^2}_{mn}(t)&=\frac{\langle u^2(t),\chi_{mn}\rangle}{\|\chi_{mn}\|_0^2}\\ &=\frac{1}{\|\chi_{mn}\|_0^2}\left\langle\sum_{p,q}\hat{u}_{pq}(t)\chi_{pq}\cdot\sum_{k,s}\hat{u}_{ks}(t)\chi_{ks},\chi_{mn}\right\rangle\\ &=\sum_{p,q,k,s}{}^{*}b(m,n;\,p,q,k,s)\hat{u}_{pq}(t)\hat{u}_{ks}(t),\end{aligned}\tag{3.13}$$

the sum with an asterisk is defined by (3.9) and the coefficients $b(m,n;\,p,q,k,s)$ are defined by (3.5). Here we have used the simple orthogonality relation

$$\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(p+k-m)}\,\mathrm{d}\theta=\begin{cases}2\pi, & p+k=m,\\ 0, & p+k\neq m.\end{cases}$$

We expand the source term into the eigenfunction series

$$f(r,\theta,t)=\sum_{m,n}\hat{f}_{mn}(t)\chi_{mn}(r,\theta),\tag{3.14}$$

where the coefficients $\hat{f}_{mn}(t)$ are defined by the (3.3)-type formula, and substitute (2.2), (3.14) and (3.13) into (2.1). Then we obtain the following initial-value problem for the coefficients $\hat{u}_{mn}(t)$ with $m \in \mathbf{Z}$, $n \in \mathbf{N}$

$$\begin{aligned} &\hat{u}''_{mn}(t) + 2b\lambda_{mn}^2 \hat{u}'_{mn}(t) + (\alpha\lambda_{mn}^4 + \lambda_{mn}^2)\hat{u}_{mn}(t) = -\beta\lambda_{mn}^2 \widehat{u^2}_{mn}(t) + a\hat{f}_{mn}(t), \quad t > 0, \\ &\hat{u}_{mn}(0) = \hat{u}'_{mn}(0) = 0. \end{aligned} \tag{3.15}$$

Integrating (3.15) with respect to t we deduce the nonlinear integral equation

$$\begin{aligned} \hat{u}_{mn}(t) = &\frac{a}{\sigma_{mn}} \int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)} \sin(\sigma_{mn}(t-\tau)) \hat{f}_{mn}(\tau)\,\mathrm{d}\tau \\ &- \frac{\beta\lambda_{mn}^2}{\sigma_{mn}} \int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)} \sin(\sigma_{mn}(t-\tau)) \widehat{u^2}_{mn}(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3.16}$$

where

$$\sigma_{mn} = \lambda_{mn}\sqrt{\mu\lambda_{mn}^2 + 1} \quad \text{and} \quad \mu = \alpha - b^2 > 0.$$

For solving (3.16) we employ the perturbation theory. We seek its solution in the form

$$\hat{u}_{mn}(t) = \sum_{N=0}^{\infty} a^{N+1} \hat{v}_{mn}^{(N)}(t). \tag{3.17}$$

Substituting (3.17) into Eq. (3.16) we obtain the recursion formulas (3.7) and (3.8) for the coefficients $\hat{v}_{mn}^{(N)}(t)$. In order to guarantee the absolute and uniform convergence of the series (3.17) we choose $a < (c_0 C_*)^{-1}$, where c_0 and C_* are the constants defined by (3.12). Thus, the solution in question is constructed in the form (2.2), (3.17), (3.7) and (3.8).

Next, we establish that the above constructed solution belongs to the required function space. By Lemma 1,

$$|\hat{f}_{mn}(t)| \leqslant f_*\sqrt{\lambda_{mn}}, \quad \text{where } f_* = \frac{\|f\|_{C_b(\mathbf{R}^+, L_2(\Omega))}}{\sqrt{2\pi\mathscr{C}_1}}$$

and $\mathscr{C}_1$ is the constant from the inequality (2.6). Applying Lemma 5 with $k = 0$, we deduce that for $m \in \mathbf{Z}$, $n \geqslant 1$, $t \geqslant 0$ and $N \geqslant 1$

$$|\hat{v}_{mn}^{(N)}(t)| \leqslant \frac{C_*^N c_0^{N+1}}{(N+1)^2} \frac{R(m)}{\lambda_{mn}^{3/2}}.$$

Consequently,

$$|\hat{u}_{mn}(t)| \leqslant cR(m)\lambda_{mn}^{-3/2}. \tag{3.18}$$

Recalling the definition of the norm in $H^s(\Omega)$ (2.6) and (2.7) we can write that

$$\|u(\cdot, t)\|_s^2 = \sum_{m,n} \lambda_{mn}^{2s} |\widehat{u}_{mn}(t)|^2 \|\chi_{mn}\|_0^2. \tag{3.19}$$

For proving the convergence of this series we consider its remainder

$$\sum_{m=M_0}^{\infty} \sum_{n=N_0}^{\infty} \lambda_{mn}^{2s} |\widehat{u}_{mn}(t)|^2 \|\chi_{mn}\|_0^2, \tag{3.20}$$

where the constants M_0 and N_0 are sufficiently large, take into account the estimates (2.6), (2.7) and compare (3.20) with the integral

$$\int_{M_0}^{\infty} \frac{\ln^2 m}{m^2} \int_{N_0}^{\infty} \frac{\mathrm{d}n}{(m+2n)^{4-2s}}. \tag{3.21}$$

Convergence of the inner integral provides the restriction $s < \frac{3}{2}$. Consequently, series (3.19) converges uniformly with respect to $t \geqslant 0$ for $s < \frac{3}{2}$. Therefore, $u(t) \in H_0^s(\Omega)$ for all $t \geqslant 0$ and $u \in C_b(\mathbf{R}^+, H_0^s(\Omega))$ for $s < \frac{3}{2}$.

(ii) *Uniqueness*: Assume that there exist two mild solutions $u^{(1)}$ and $u^{(2)}$ from the space $C_b(\mathbf{R}^+, H_0^s(\Omega))$ with $s < \frac{3}{2}$. Then each of them can be expanded into the series (3.1), where the coefficients $u_{mn}^{(1)}$ and $u_{mn}^{(2)}$ satisfy the integral equation (3.16) and have the estimate (3.18). Setting $w = u^{(1)} - u^{(2)}$ we can expand it into the series of the type of (2.2) and get

$$w = \sum_{m,n} \hat{w}_{mn}(t)\chi_{mn}(r,\theta),$$

where

$$\hat{w}_{mn}(t) = -\frac{\beta\lambda_{mn}^2}{\sigma_{mn}}\int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)}\sin(\sigma_{mn}(t-\tau))\left[\widehat{(u^{(1)})^2}_{mn}(\tau) - \widehat{(u^{(2)})^2}_{mn}(\tau)\right]\mathrm{d}\tau.$$

Here

$$\begin{aligned}\widehat{(u^{(1)})^2}_{mn}(t) - \widehat{(u^{(2)})^2}_{mn}(t) &= \sum_{p,q,k,s}{}^{*} b(m,n;\,p,q,k,s)\left[\hat{u}_{pq}^{(1)}(t)\hat{w}_{ks}(t) + \hat{w}_{pq}(t)\hat{u}_{ks}^{(2)}(t)\right]\\ &= \left(\sum_{\substack{p,q,k,s:p+k=m\\ \lambda_{pq}>\lambda_{ks}}} + \sum_{\substack{p,q,k,s:p+k=m\\ \lambda_{pq}<\lambda_{ks}}} + \sum_{\substack{p,q,k,s:p+k=m\\ \lambda_{pq}=\lambda_{ks}}}\right)\\ &\quad\times b(m,n;\,p,q,k,s)\left[\hat{u}_{pq}^{(1)}(t)\hat{w}_{ks}(t) + \hat{w}_{pq}(t)\hat{u}_{ks}^{(2)}(t)\right]\\ &= S_1 + S_2 + S_3.\end{aligned}$$

Choosing any $\delta > 0$ and applying Lemma 3 and (2.6) we can write the following chain of inequalities for the sum S_1

$$\begin{aligned}S_1 &\leqslant C\sqrt{\lambda_{mn}} \sum_{\substack{p,q,k,s:p+k=m\\ \lambda_{pq}>\lambda_{ks}}} \frac{|\hat{u}_{pq}^{(1)}(t)|}{\lambda_{pq}^{3/2}}\frac{|\hat{w}_{ks}(t)|}{\lambda_{ks}^{1/2}}\\ &\leqslant C\sqrt{\lambda_{mn}} \sum_{\substack{p,q,k,s:p+k=m\\ \lambda_{pq}>\lambda_{ks}}} \frac{q^{(1+\delta)/2}|\hat{u}_{pq}^{(1)}(t)|}{\lambda_{pq}^{3/2}\lambda_{ks}^{\rho}}\cdot\frac{|\hat{w}_{ks}(t)|\lambda_{ks}^{\rho}}{\lambda_{ks}^{1/2}q^{(1+\delta)/2}}.\end{aligned}$$

Using Cauchy–Schwartz inequality we get

$$\begin{aligned}S_1 &\leqslant C\sqrt{\lambda_{mn}} \sum_{\substack{q,k,s:\\ \lambda_{pq}>\lambda_{ks}}} \frac{q^{(1+\delta)/2}|\hat{u}_{pq}^{(1)}(t)|}{\lambda_{m-k,q}^{3/2}\lambda_{ks}^{\rho}}\cdot\frac{|\hat{w}_{ks}(t)|\lambda_{ks}^{\rho}}{\lambda_{ks}^{1/2}q^{(1+\delta)/2}}\\ &\leqslant C\sqrt{\lambda_{mn}}\left(\sum_{q,k,s}\frac{q^{1+\delta}}{\lambda_{m-k,q}^{6}\lambda_{ks}^{2\rho}}\right)^{1/2}\left(\sum_{q=1}^{\infty}\frac{1}{q^{1+\delta}}\right)^{1/2}\left(\sum_{k,s}\lambda_{ks}^{2\rho}|\hat{w}_{ks}(t)|^2\|\chi_{ks}\|^2\right)^{1/2}\\ &\leqslant C\sqrt{\lambda_{mn}}\|w(t)\|_{\rho}^{2}.\end{aligned}$$

It follows from (2.7) that the first series in q, k, s converges provided that $\rho > -1 + \delta$. Since δ can be arbitrarily small it implies that $\rho > -1$. The sum S_2 can be estimated in a similar way. The sum S_3 is even easier to estimate since the condition $\lambda_{pq} = \lambda_{ks}$ implies that $p = k$ and $q = s$.

Thus, we deduce that

$$|\hat{w}_{mn}(t)|^2 \leqslant C\lambda_{mn}\left(\int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)}\|w(\tau)\|_{\rho}\,\mathrm{d}\tau\right)^2.$$

Multiplying both sides of the last inequality by $\lambda_{mn}^{2\rho}\|\chi_{mn}\|^2$ and summing in m and n we deduce that for some $T>0$, $t\in[0,T]$ and $\rho<1$

$$\sup_{t\in[0,T]}\|w(t)\|_\rho^2 \leqslant C\Phi(T)\left(\sup_{t\in[0,T]}\|w(t)\|_\rho\right)^2,$$

where

$$\Phi(t)=\sum_{m,n}\lambda_{mn}^{2\rho}\|\chi_{mn}\|^2\left(\int_0^t \mathrm{e}^{-b\lambda_{mn}^2(t-\tau)}\,\mathrm{d}\tau\right)^2 = \sum_{m,n}\lambda_{mn}^{2\rho}\|\chi_{mn}\|^2\left[\frac{1-\mathrm{e}^{-b\lambda_{mn}^2 t}}{b\lambda_{mn}^2}\right]^2.$$

The last series converges absolutely and uniformly with respect to $t\geqslant 0$ for all $\rho<\frac{3}{2}$. Consequently, $\Phi(t)$ is a non-decreasing continuous function on $[0,T]$ and $\Phi(0)=0$. Therefore

$$\left(\sup_{t\in[0,T]}\|w(t)\|_\rho\right)^2 \leqslant C(T)\left(\sup_{t\in[0,T]}\|w(t)\|_\rho\right)^2,$$

where the constant $C(T)=C\Phi(T)$ can be made less than one by the appropriate choice of T. This contradiction allows to prove uniqueness for the interval $[0,T]$.

Next, we extend this result to $t\in\mathbf{R}$. Choosing the intervals $[T_k,T_{k+1}]$, $k=1,2,\ldots,$ with $T_k=kH$ for $k\to\infty$ and taking into account that

$$\int_{T_k}^{T_{k+1}}\mathrm{e}^{-b\lambda_{mn}^2\tau}\,\mathrm{d}\tau=\frac{1-\mathrm{e}^{-b\lambda_{mn}^2 H}}{b\lambda_{mn}^2}\mathrm{e}^{-b\lambda_{mn}^2 T_k}$$

one can note that

$$\left(\sup_{t\in[T_k,T_{k+1}]}\|w(t)\|_\rho\right)^2 \leqslant C\Phi(H)\left(\sup_{t\in[T_k,T_{k+1}]}\|w(t)\|_\rho\right)^2$$

with the condition $C\Phi(H)<1$ satisfied earlier. This contradiction allows one to establish uniqueness for all $t\geqslant 0$ and $\rho<\frac{3}{2}$ and completes the proof of Theorem 1. □

4. Long-time asymptotics

In this section we provide an important example of computing long-time asymptotics of the solution in question for a special choice of the forcing term, $f(r,\theta,t)=F(r,\theta)\cos(\omega t)\mathrm{e}^{-\kappa t}$. This term provides an example of separation of variables and serves as a typical acoustic signal.

Theorem 3. *If* $\alpha>b^2$, $f(r,\theta,t)=F(r,\theta)\cos(\omega t)\mathrm{e}^{-\kappa t}$, *where* $F\in L_2(\Omega)$, $\kappa,\ \omega=const>0$, $0<2\kappa<b\lambda_{01}^2$ *and* $a<(c_0C_*)^{-1}$(*see* (3.12)), *then the estimate holds for* $t>0$ *and* $s<\frac{3}{2}$

$$\|u(\cdot,t)-U(\cdot,t)\|_s\leqslant C\mathrm{e}^{-2\kappa t}, \tag{4.1}$$

where

$$U(r,\theta,t)=\sum_{m,n}U_{mn}(t)J_m(\lambda_{mn}r)\mathrm{e}^{\mathrm{i}m\theta},$$

$$U_{mn}(t)=\frac{a}{2}\mathrm{e}^{-\kappa t}\left\{[I_{c,mn}^+(t)-I_{c,mn}^-(t)]\frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}-[I_{s,mn}^+(t)-I_{s,mn}^-(t)]\frac{\cos(\sigma_{mn}t)}{\sigma_{mn}}\right\}\hat{F}_{mn},$$

$$\hat{F}_{mn} = \frac{1}{2\pi\|J_m(\lambda_{mn}\cdot)\|^2}\int_{-\pi}^{\pi}\int_0^1 F(r,\theta)J_m(\lambda_{mn}r)\mathrm{e}^{-\mathrm{i}m\theta}r\,\mathrm{d}r\,\mathrm{d}\theta,$$

$$I^+_{c,mn}(t) = \frac{(b\lambda^2_{mn}-\kappa)\cos[(\sigma_{mn}+\omega)t] + (\sigma_{mn}+\omega)\sin[(\sigma_{mn}+\omega)t]}{(b\lambda^2_{mn}-\kappa)^2 + (\sigma_{mn}+\omega)^2},$$

$$I^-_{c,mn}(t) = \frac{(b\lambda^2_{mn}-\kappa)\cos[(\sigma_{mn}-\omega)t] + (\sigma_{mn}-\omega)\sin[(\sigma_{mn}-\omega)t]}{(b\lambda^2_{mn}-\kappa)^2 + (\sigma_{mn}-\omega)^2},$$

$$I^+_{s,mn}(t) = \frac{(b\lambda^2_{mn}-\kappa)\sin[(\sigma_{mn}+\omega)t] - (\sigma_{mn}+\omega)\cos[(\sigma_{mn}+\omega)t]}{(b\lambda^2_{mn}-\kappa)^2 + (\sigma_{mn}+\omega)^2},$$

$$I^-_{s,mn}(t) = \frac{(b\lambda^2_{mn}-\kappa)\sin[(\sigma_{mn}-\omega)t] - (\sigma_{mn}-\omega)\cos[(\sigma_{mn}-\omega)t]}{(b\lambda^2_{mn}-\kappa)^2 + (\sigma_{mn}-\omega)^2}$$

and $\mu = \alpha - b^2 > 0$. *Here the constant* $C = C(\alpha, b, \beta)$ *is independent of* t.

Proof. It is easy to see that the chosen source term satisfies the assumptions of Theorem 1. Therefore, the solution in question can be represented in the form

$$u = U(r,\theta,t) + W(r,\theta,t),$$

where

$$U(r,\theta,t) = a\sum_{m,n}\hat{v}^{(0)}_{mn}(t)\chi_{mn}(r,\theta),$$

$$W(r,\theta,t) = \sum_{m,n}W_{mn}(t)\chi_{mn}(r,\theta)$$

and

$$W_{mn}(t) = \sum_{N=1}^{\infty}a^{N+1}\hat{v}^{(N)}_{mn}(t).$$

The proof of Lemma 4 provides understanding of the structure of the perturbation series representing Fourier coefficients of the solution $\hat{u}_{mn}(t)$. Indeed, the coefficients $\hat{v}^{(0)}_{mn}(t)$ satisfy inequality (3.10) and the coefficients $\hat{v}^{(N)}_{mn}(t)$ satisfy estimates (3.11). Consequently, the function U forms the major part of the long-time asymptotics and the function R satisfies the estimate for $s < \frac{3}{2}$

$$\|W(\cdot,t)\|_s \leqslant C\mathrm{e}^{-2\kappa t},$$

and forms its residual term. Here the constant C is independent of r, θ and t.

It remains to compute $\hat{v}^{(0)}_{mn}(t)$. According to (3.7), they are

$$\hat{v}^{(0)}_{mn}(t) = \hat{F}_{mn}\mathrm{e}^{-b\lambda^2_{mn}t}\left[\frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}\int_0^t \mathrm{e}^{(b\lambda^2_{mn}-\kappa)\tau}\cos(\sigma_{mn}\tau)\cos(\omega\tau)\,\mathrm{d}\tau - \frac{\cos(\sigma_{mn}t)}{\sigma_{mn}}\int_0^t \mathrm{e}^{(b\lambda^2_{mn}-\kappa)\tau}\sin(\sigma_{mn}\tau)\cos(\omega\tau)\,\mathrm{d}\tau\right]. \quad (4.2)$$

Note that here $b\lambda_{mn}^2 - \kappa \geqslant b\lambda_{01}^2 - \kappa > 0$. Taking into account two simple results

$$I_c(t) = \int_0^t e^{A\tau} \cos(B\tau)\, d\tau = \frac{e^{At}[A\cos(Bt) + B\sin(Bt)] - A}{A^2 + B^2},$$
$$I_s(t) = \int_0^t e^{A\tau} \sin(B\tau)\, d\tau = \frac{e^{At}[A\sin(Bt) - B\cos(Bt)] + B}{A^2 + B^2},$$

computing the integrals in (4.2) and leaving the terms of order $O(e^{-\kappa t})$ we deduce (4.1). The proof is complete. □

Remark 3. We have used the assumption $0 < 2\kappa < b\lambda_{01}^2$ to establish some "order" with respect to decay rates of the source term and the exponentials $e^{-b\lambda_{mn}^2 t}$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$. If another "ordering" relation holds, e.g., $0 < \kappa < b\lambda_{01}^2 < 2\kappa < b\lambda_{11}^2$, then it follows from the proof of Lemma 4 that the asymptotic estimate is

$$\|u(\cdot,t) - U(\cdot,t)\|_s \leqslant C e^{-b\lambda_{01}^2 t}.$$

In a similar way it is possible to deal with other relations between κ and λ_{mn}.

Remark 4. The next terms of the long-time asymptotic expansion can be calculated on the basis of the major term if one uses the recursion relations (3.7), (3.8).

5. Numerical simulations

In this section we use the eigenfunction method of Section 3 for numerical simulations. In fact we use a straightforward numerical adaptation of this approach. In our future work we will optimize the numerical method for speed and accuracy. It will also be the topic of future research to compare the convergence properties of the algorithm in question with those of other numerical methods. However, we would like to emphasize already here that a unique property of the eigenfunction method (compared to others) is that (3.1) along with (3.18), (3.19), (3.21), and (2.7) provides us with a better understanding of the convergence rate of approximations for the *nonlinear* problem (2.1). Namely,

$$\|u(\cdot,t)\|_s^2 \leqslant C \sum_{m,n} R^2(m) \lambda_{mn}^{2s-4} = \mathcal{O}\left(\sum_m \frac{\ln^2 m}{m^2} \sum_n \frac{1}{(m+2n)^{4-2s}} \right).$$

We used FORTRAN95 with IMSL and CERNLIB subroutines for numerical calculations, and MATLAB 6.5 for visualization on a LINUX workstation with 2 × 2 GHz processors and 2 GB memory.

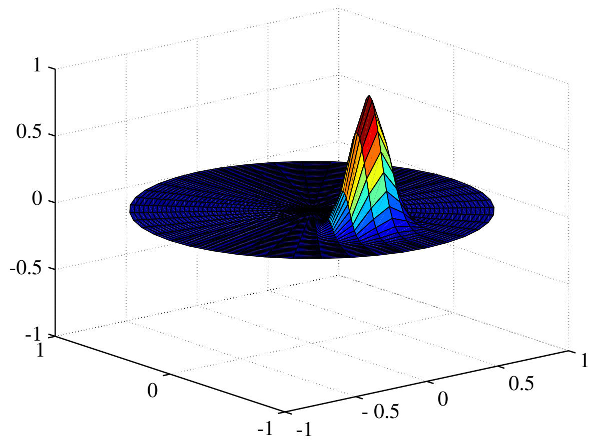

Fig. 1. Forcing functions at time $t = 0$.

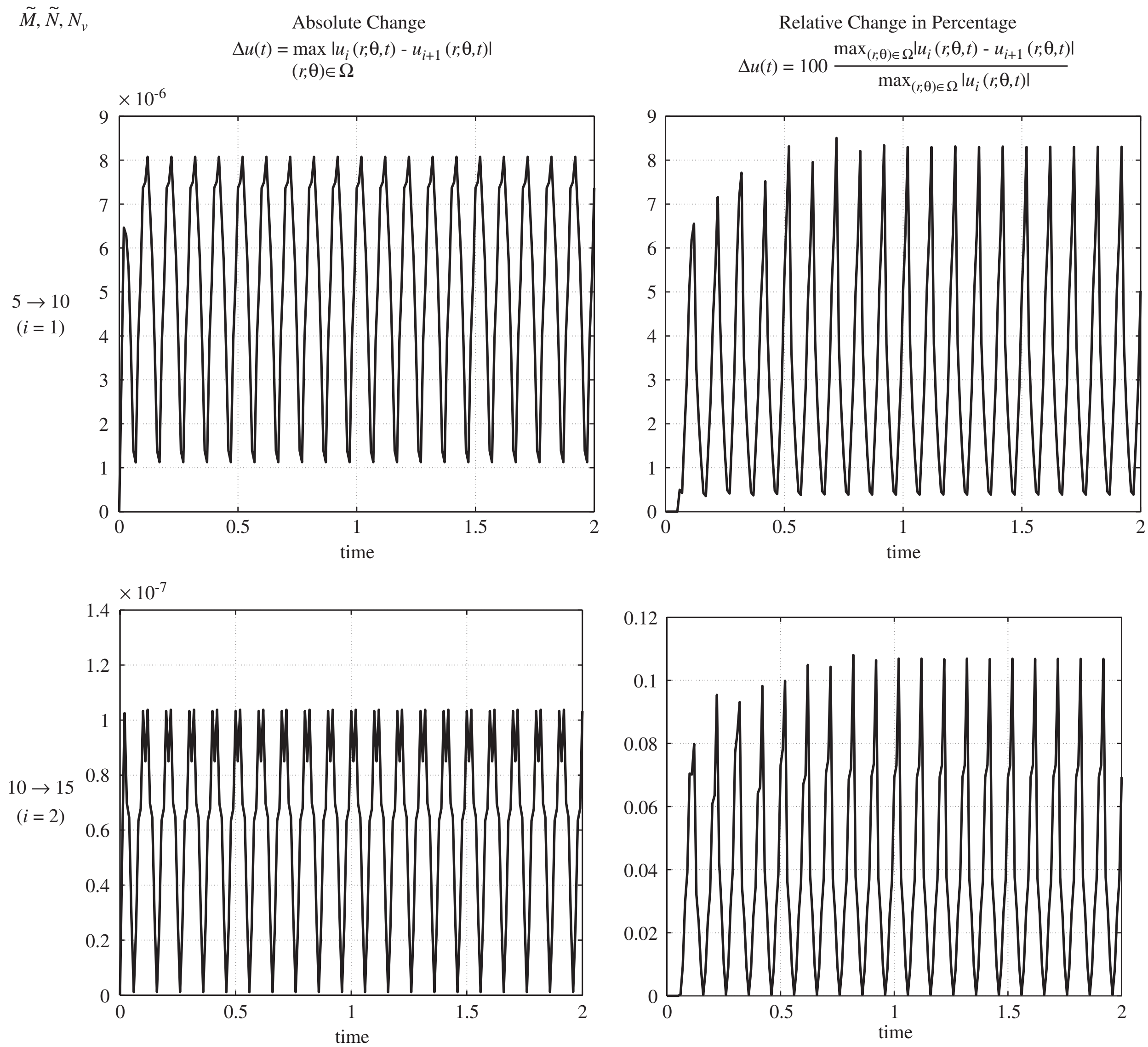

Fig. 2. Maximum change in the approximations as $\widetilde{M}$, $\widetilde{N}$ and N_v are increased from 5 to 10 and from 10 to 15 ($\kappa = 0$).

5.1. The numerical procedure

The numerical procedure, based on (3.1), (3.17), (3.7) and (3.8) consists of the following main steps:

1. Calculate the Fourier–Bessel coefficients of the forcing term using the formula

$$\hat{f}_{mn}(t) = \frac{\langle f, \chi_{mn}\rangle}{\|\chi_{mn}\|_0^2} = \frac{\int_0^1 \int_0^{2\pi} f(r,\theta,t) J_m(\lambda_{mn} r) \mathrm{e}^{-\mathrm{i}m\theta} r \,\mathrm{d}\theta\,\mathrm{d}r}{\int_0^1 \int_0^{2\pi} |\chi_{mn}(r,\theta)|^2 r \,\mathrm{d}\theta\,\mathrm{d}r}, \tag{5.1}$$

where $m = -\widetilde{M}, \ldots, \widetilde{M}$, and $n = 1, \ldots, \widetilde{N}$ for given $\widetilde{M}, \widetilde{N}, \in \mathbf{N}$. The integrations in polar coordinates require discretization. We used simple Riemann sums in polar coordinates with constant $\mathrm{d}r$ and $\mathrm{d}\theta$ mesh sizes.

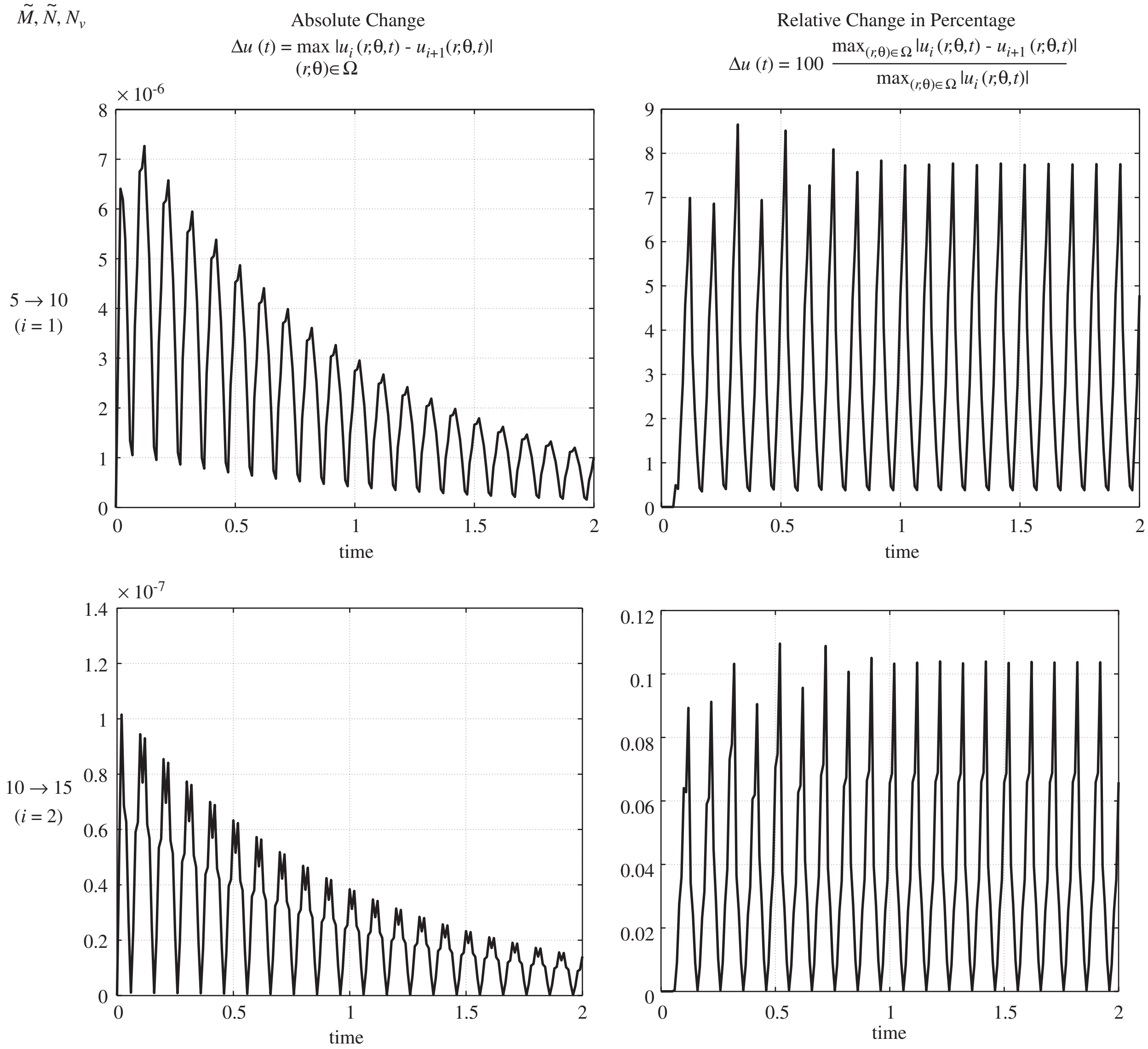

Fig. 3. Maximum change in the approximations as $\widetilde{M}$, $\widetilde{N}$ and N_v are increased from 5 to 10 and from 10 to 15 ($\kappa = 1$).

2. Calculate the nonlinear coefficients

$$
\begin{aligned}
b(m,n;\, p,q,k,s) &= \frac{\langle \chi_{pq} \cdot \chi_{ks}, \chi_{mn} \rangle}{\|\chi_{mn}\|_0^2} \\
&= \frac{\int_0^1 \int_0^{2\pi} J_p(\lambda_{pq} r) J_k(\lambda_{ks} r) J_m(\lambda_{mn} r) \exp(\mathrm{i}(p+k-m)\theta) r \,\mathrm{d}\theta \,\mathrm{d}r}{\int_0^1 \int_0^{2\pi} |\chi_{mn}(r,\theta)|^2 r \,\mathrm{d}\theta \,\mathrm{d}r}
\end{aligned}
\tag{5.2}
$$

where $m = -\widetilde{M}, \ldots, \widetilde{M}$, $n = 1, \ldots, \widetilde{N}$. While the total number of terms is $(\widetilde{M}+1)^3 \widetilde{N}^3$, due to orthogonality $b(m,n;\, p,q,k,s) \neq 0$ only if $p + k = m$. This reduces the number of nonzero terms to $(\widetilde{M}+1)^2 \widetilde{N}^3$. Further reduction in storage can be achieved by using the symmetry of the terms.

3. Calculate the perturbation series recursively

$$\hat{v}_{mn}^{(0)}(t) = \int_0^t e^{-b\lambda_{mn}^2(t-\tau)} \frac{\sin[\sigma_{mn}(t-\tau)]}{\sigma_{mn}} \hat{f}_{mn}(\tau)\,d\tau, \tag{5.3}$$

$$\hat{v}_{mn}^{(l)}(t) = -\frac{\beta\lambda_{mn}^2}{\sigma_{mn}} \int_0^t e^{-b\lambda_{mn}^2(t-\tau)} \sin[\sigma_{mn}(t-\tau)] \times \sum_{p,q,k,s}{}^{*} b(m,n,p,q,k,s) \sum_{j=1}^{l} \hat{v}_{pq}^{(j-1)}(\tau)\hat{v}_{ks}^{(l-j)}(\tau)\,d\tau \tag{5.4}$$

for $l=1,\ldots,N_v$, $t\in[0,T]$, $m=-\widetilde{M},\ldots,\widetilde{M}$ and $n=1,\ldots,\widetilde{N}$. This part is the most demanding computationally. It requires the evaluation of convolution type integrals and four sums embedded in the integral. We used Simpson's method for approximation of the convolution with constant time step size dt.

4. Assemble approximation of the Fourier–Bessel coefficients

$$\hat{u}_{mn}(t) \approx \sum_{l=0}^{Nv} a^{l+1}\hat{v}_{mn}^{(l)}(t) \tag{5.5}$$

at all gridpoints t. This part is trivial.

5. Assemble approximation of the solution

$$u(r,\theta,t) \approx \sum_{m=0}^{\widetilde{M}} \sum_{n=1}^{\widetilde{N}} \hat{u}_{mn}(t) J_m(\lambda_{mn} r) e^{im\theta} \tag{5.6}$$

at gridpoints t, r, θ.

Certain values related to special functions can be calculated in advance. One of these quantities is the truncated set of zeros for the Bessel functions: $\{\lambda_{mn}\}_{m,n=0,1}^{\widetilde{M},\widetilde{N}}$. In order to approximate these zeros we used the CERNLIB subroutine DBZEJY, which is based on an iterative method by Temme [25]. Another sequence of quantities consists of the squared L_2-norms of the eigenfunctions

$$\{\|\chi_{mn}\|_0^2\}_{m,n=-\widetilde{M},1}^{\widetilde{M},\widetilde{N}} = \left\{2\pi \int_0^1 r J_m^2(\lambda_{mn} r)\,dr\right\}_{m,n=-\widetilde{M},1}^{\widetilde{M},\widetilde{N}} \tag{5.7}$$

which has the symmetry property $\|\chi_{mn}\|_0 = \|\chi_{-mn}\|_0$.

5.2. Discretization errors and convergence properties

A clear advantage of this method is that there is no modeling error involved. Errors are arising only in the discretization steps and from the truncations of infinite series. Our approach has several of both. A distinctive property of the method is that by working in the Fourier–Bessel space the spatial discretization appears only in the initial (5.1), (5.2), (5.7) and final (5.6) phases of the algorithm. Also, there is no explicit numerical differentiation involved.The forcing term is the spatially localized function

$$f(r,\theta,t) = \exp\{-50[\tfrac{1}{4} + r^2 - r\cos(\theta)]\}\cos(10\pi t)\exp(-\kappa t) \tag{5.8}$$

for $t \geqslant 0$, $r \in [0,1]$ and $\theta \in [0,2\pi]$, depicted in Fig. 1 for the case $\kappa = 0$. We also use the value $\kappa = 1$, in which case the forcing term decays exponentially in time. The other values of parameters used in this section are $\alpha = 1$, $\beta = 1$, and $b = 0.9$. We estimate the perturbation parameter a of Theorem 1 with the help of (3.12) in the

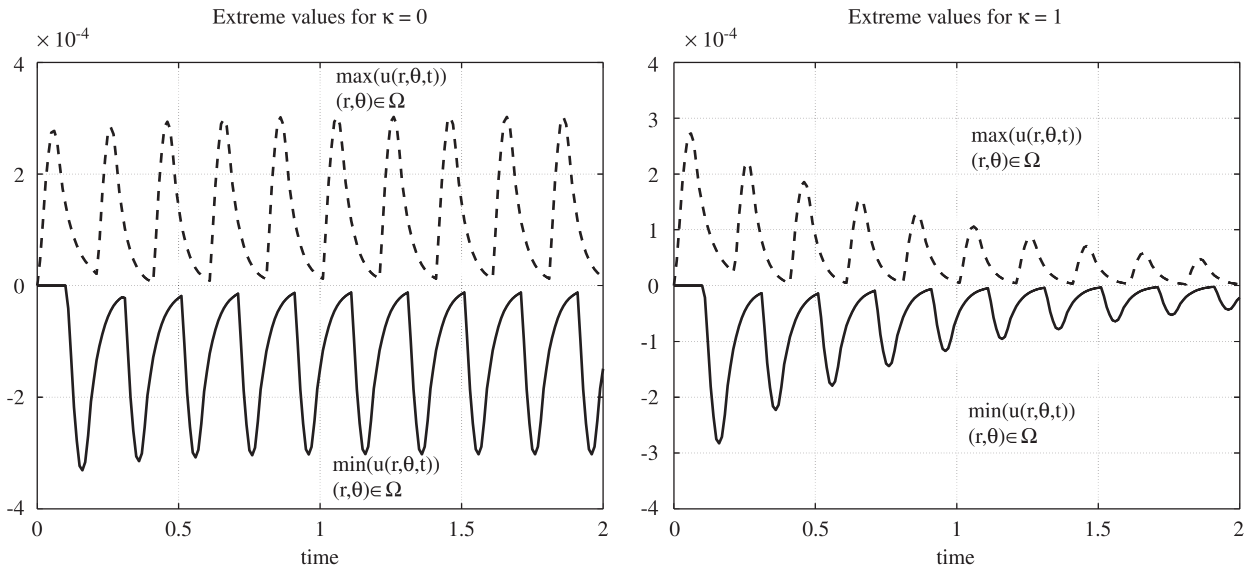

Fig. 4. Extreme values for non-decaying ($\kappa = 0$) and decaying ($\kappa = 1$) forcing.

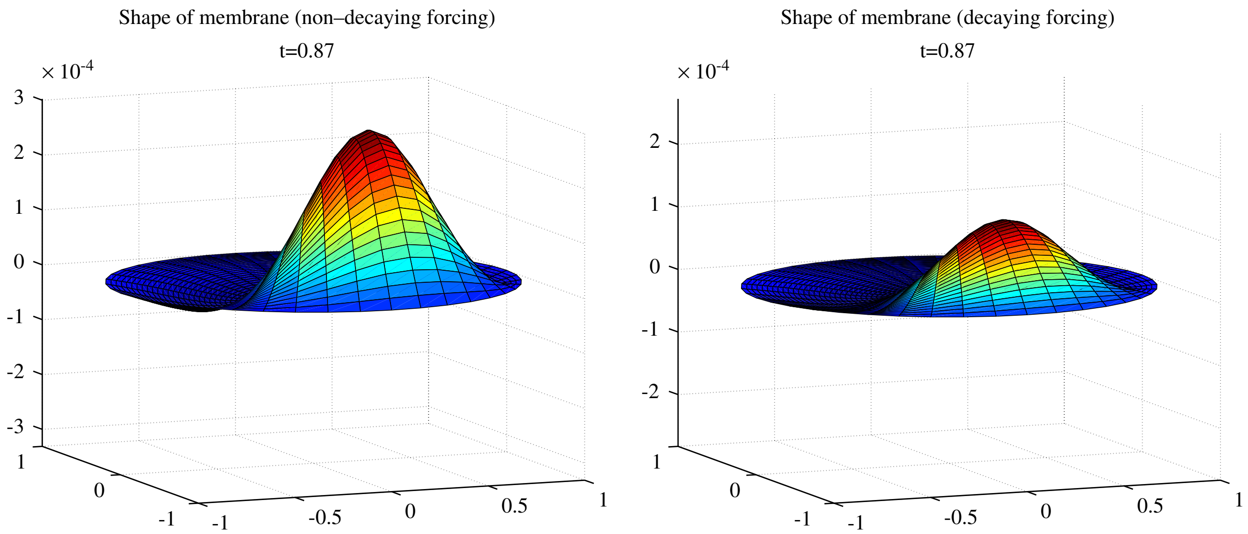

Fig. 5. Membranes at time $t = 0.87$.

following way:

1. For the case $\kappa = 0$

$$c_0 = \frac{f_*}{\sqrt{\mu}b[1 - \kappa/(b\lambda_{01}^2)]} \approx 8.440 \times 10^{-2},$$

$$C_* = \frac{2\pi^2|\beta|}{3\sqrt{\mu}b[1 - 2\kappa/(b\lambda_{01}^2)]} \approx 1.677 \times 10^{1},$$

$$a < 1/(c_0 C_*) \approx 7.064 \times 10^{-1}.$$

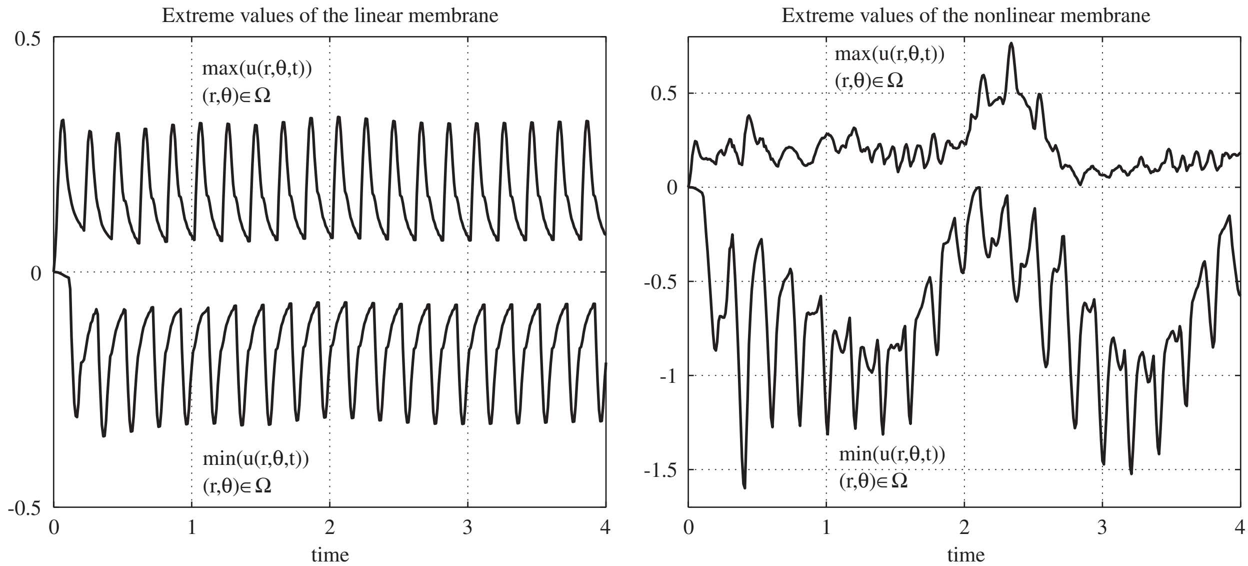

Fig. 6. Extreme values of linear and nonlinear membrane oscillations.

2. For the case $\kappa = 1$

$$c_0 = \frac{f_*}{\sqrt{\mu} b[1 - \kappa/(b\lambda_{01}^2)]} \approx 1.045 \times 10^{-1},$$

$$C_* = \frac{2\pi^2|\beta|}{3\sqrt{\mu} b[1 - 2\kappa/(b\lambda_{01}^2)]} \approx 2.724 \times 10^{1},$$

$$a < 1/(c_0 C_*) \approx 3.514 \times 10^{-1}.$$

Therefore we set $a = 0.35$ in this subsection.

For the spatial discretization we used $Nr = Nth = 40$ grid points both in the radial and the angular directions. In the time discretization $Nt = 200$ time steps were taken of the size $\mathrm{d}t = 10^{-2}$. We demonstrate briefly the convergence properties of the two partial sums (5.5) and (5.6) only. We doubled the number of zeros $\widetilde{N}$, the number of Bessel functions $\widetilde{M}$ and the number of terms in (5.5) N_v from $\widetilde{M} = \widetilde{N} = N_v = 5$ to 10 and then we further increased these values from 10 to 15. These numbers correspond to $\widetilde{M}\widetilde{N} = 25,\ 100$, and 225 number of terms in the finite sum (5.6).

Note that in the numerical application of Galerkin's method (both of spectral and of finite element type) it is usual to have a relatively low number of basis functions when one solves the second order nonlinear system of ODEs (3.15). The main reason for this is the high computational cost of solving nonlinear systems of ODEs and the usually good convergence property of Galerkin's methods observed numerically. In our case the good convergence property is established in the proof of Theorem 1.

The obtained numerical solutions are denoted by u_1, u_2, and u_3 for $\widetilde{M} = \widetilde{N} = N_v = 5$, 10 and 15, respectively. The results are summarized in Figs. 2 and 3 for the cases $\kappa = 0$ and $\kappa = 1$, respectively. In the second columns of these figures we plot the maximum change between the two numerical approximations, i.e.,

$$\Delta u(t) = \max_{(r,\theta)\in\Omega} |u_i(r,\theta,t) - u_{i+1}(r,\theta,t)|, \quad t \in [0, 2], \quad i = 1, 2.$$

In the third column of the figures we plot the relative change (in percentage) between the two numerical approximations, i.e.,

$$\Delta u(t) = 100\,\frac{\max_{(r,\theta)\in\Omega} |u_i(r,\theta,t) - u_{i+1}(r,\theta,t)|}{\max_{(r,\theta)\in\Omega} |u_i(r,\theta,t)|}, \quad t \in [0, 2], \quad i = 1, 2.$$

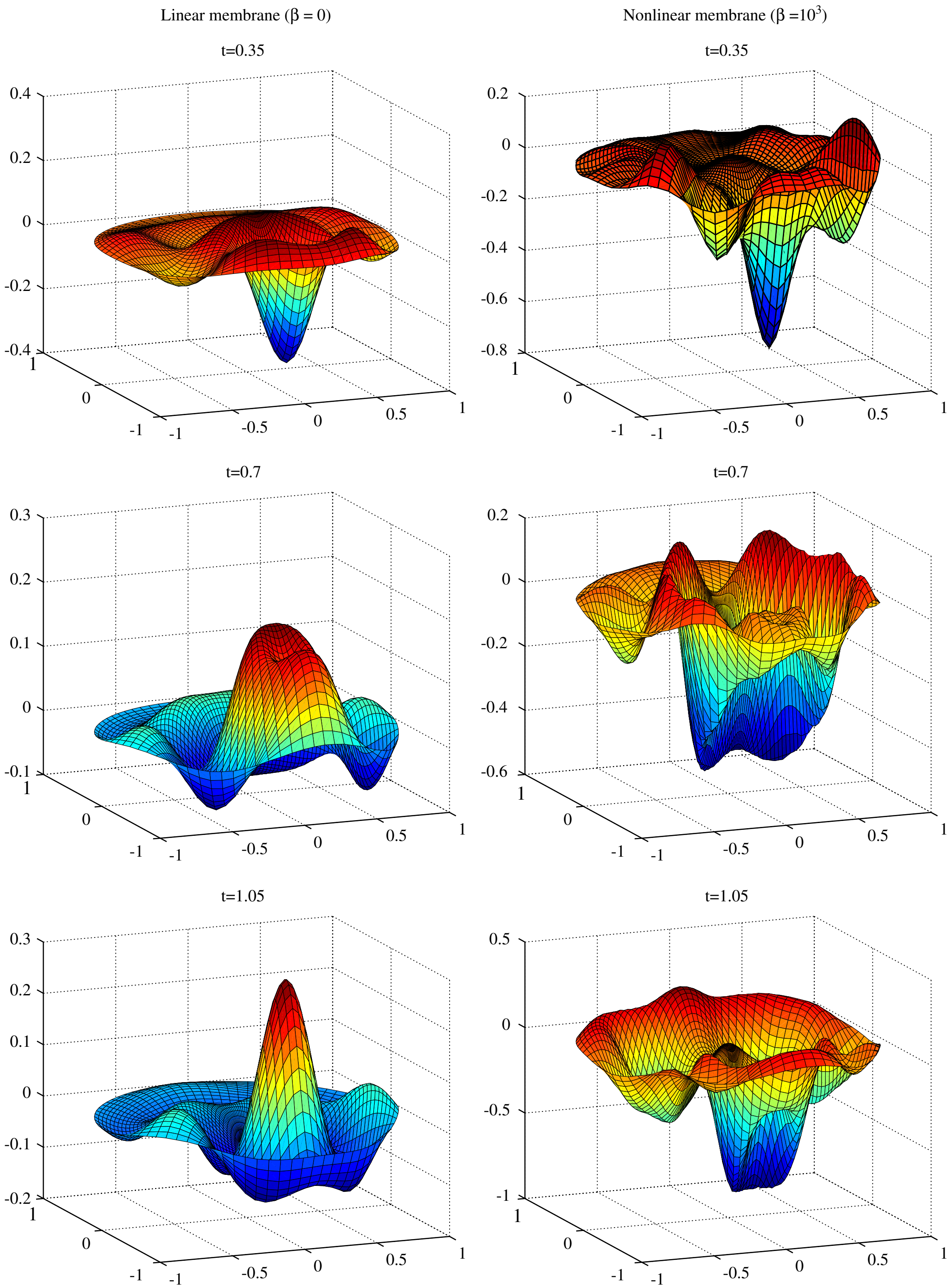

Fig. 7. Linear and nonlinear membrane oscillations, $t = 0.35, 0.7, 1.05$.

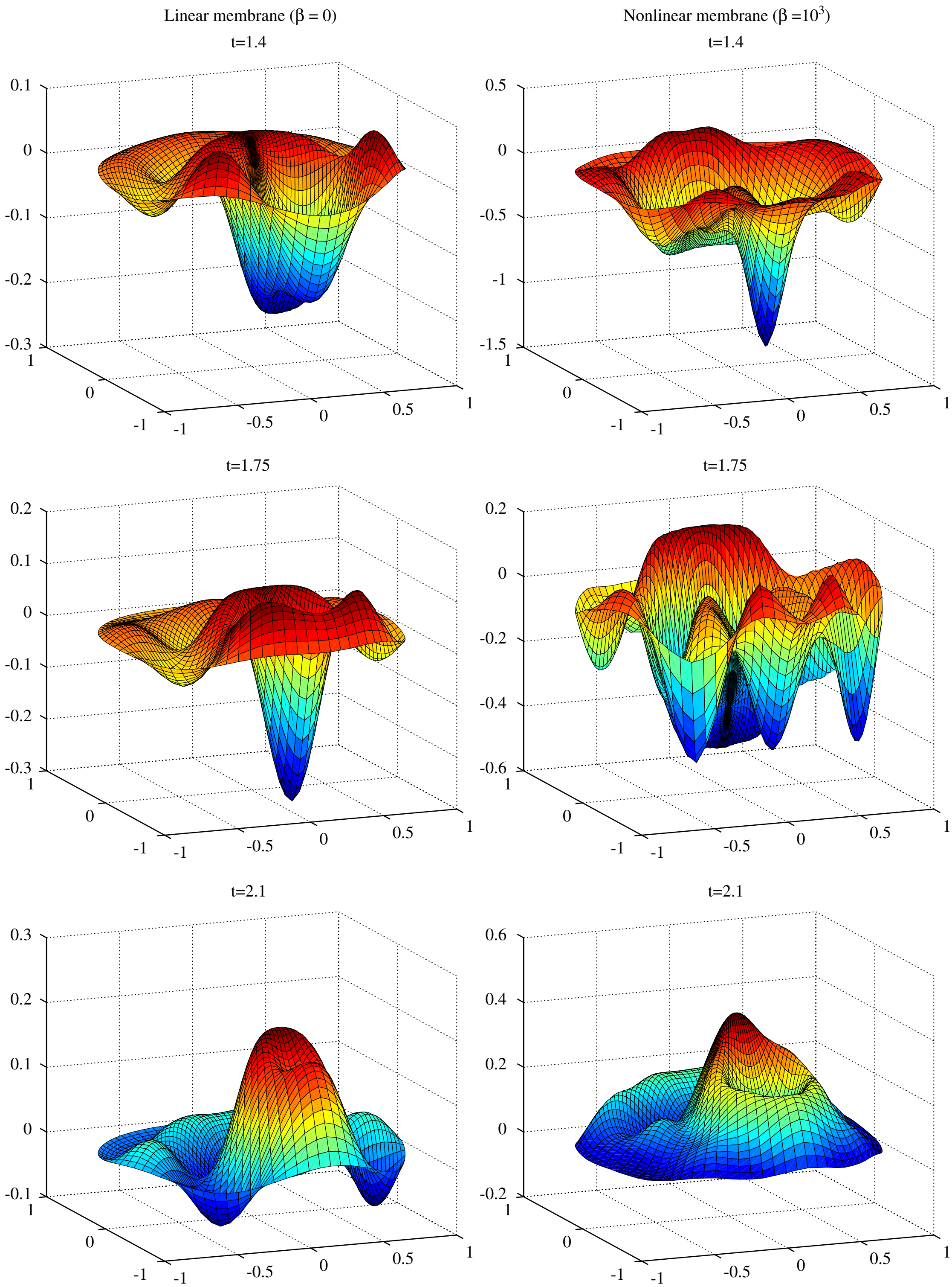

Fig. 8. Linear and nonlinear membrane oscillations, $t = 1.4, 1.75, 2.1$.

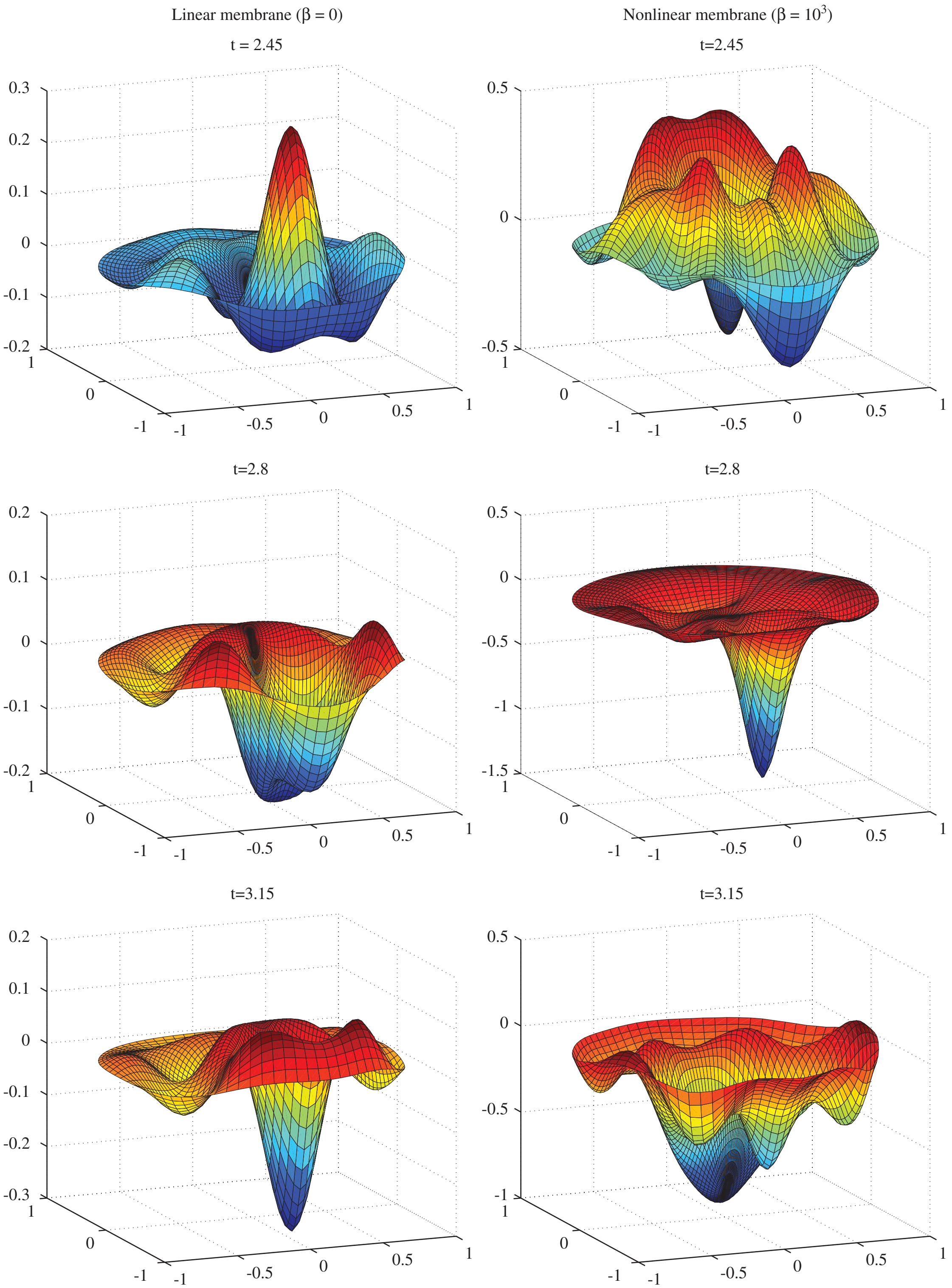

Fig. 9. Linear and nonlinear membrane oscillations, $t = 2.45, 2.8, 3.15$.

Table 1
Convergence in the series $\hat{u}_{mn}(t) = \sum_{l=0}^{Nv} a^{l+1}\hat{v}_{mn}^{(l)}(t)$

l	a^{l+1}	$\max_{m,n,t} \lvert\hat{v}_{mn}^{(l)}(t)\rvert$	$a^l \max_{m,n,t} \lvert\hat{v}_{mn}^{(l)}(t)\rvert$
0	$7.00E+02$	$1.25E-04$	$8.75E-02$
1	$4.90E+05$	$4.25E-08$	$2.08E-02$
2	$3.43E+08$	$1.46E-12$	$5.01E-04$
3	$2.40E+11$	$1.45E-16$	$3.48E-05$
4	$1.68E+14$	$1.39E-20$	$2.34E-06$
5	$1.18E+17$	$2.49E-24$	$2.93E-07$
6	$8.24E+19$	$3.05E-28$	$2.51E-08$
7	$5.77E+22$	$3.27E-32$	$1.89E-09$
8	$4.04E+25$	$4.65E-36$	$1.88E-10$
9	$2.83E+28$	$4.04E-40$	$1.41E-11$
10	$1.98E+31$	$4.49E-44$	$8.88E-13$

Since we are working with small oscillations, comparing the absolute changes of 8×10^{-6} and 10^{-7} in solutions is not descriptive. Comparing the relative changes for both $\kappa = 0$ and $\kappa = 1$ we observe in the last column of Figs. 2 and 3 almost identical features. After doubling the three discretization parameters $\widetilde{M}$, $\widetilde{N}$ and N_v from 5 to 10 there appears about 8% change in the solutions for both cases, and a further increase of the aforementioned discretization parameters from 10 to 15 results in less than 0.12% change in the solution, which we consider to be very small.

5.3. Membrane oscillation for non-decaying and for decaying forcing

In Fig. 4 we compare extreme values of the solution of system (2.1) for the cases of non-decaying and decaying forcing, respectively, as time increases from $t = 0$ to $t = 2$. When the source term is non-decaying, the maximum and minimum values of the solution remain at the constant levels of 3×10^{-4} and 3×10^{-4}, respectively (first column of Fig. 4). When the forcing decays exponentially, so do the extreme values of the solution (second column of Fig. 4). Otherwise the minimum and maximum values are reached at the same time instants in both cases. Fig. 5 shows the membrane displacements at $t = 0.87$ for $\kappa = 0$ and $\kappa = 1$, respectively. The shapes of the two membranes are very similar, only the deviations are smaller for decaying forcing.

5.4. Nonlinear oscillations

We emphasize that (2.1) models *small* nonlinear oscillations of certain uniform elastic membranes. However, in this section we use parameter values for which oscillations are relatively large, and hence the nonlinear term has a prominent effect on the solution. The parameter values are now $\alpha = 0.1$, $a = 700$ and $b = 0.1$. The damping is small and oscillations are strong.

We present a comparison of the linear ($\beta = 0$) and strongly nonlinear ($\beta = 10^3$) membrane oscillations under forcing (5.8). In Fig. 6 we plotted the maximum vertical deflection of the membrane under and above the horizontal plane for the linear and nonlinear cases, respectively. The extreme values of the solution are vertically symmetric and oscillations seem to be periodic in time for the linear membrane. In the nonlinear case the extreme values are skewed downward and there is no time periodicity.

Figs. 7–9 shows the linear and nonlinear membranes at various time instants. The strong nonlinear effects are clearly visible in the second column of the table. An interesting consequence of the periodic forcing in the linear case is that oscillations seem to be periodic with the shape of the membrane being the same for $t = 1.75$ and $t = 3.15$ as well as for $t = 1.05$ and $t = 2.45$. We did not observe such periodicity in the nonlinear case.

The sequence of partial sums

$$\hat{u}_{mn}(t) = \sum_{l=0}^{N_v} a^{l+1}\hat{v}_{mn}^{(l)}(t) \tag{5.9}$$

approximating the Fourier–Bessel coefficients shows good convergence properties even for the case of a large perturbation parameter a (e.g., $a = 700$, see Table 1). Therefore a typical computational difficulty of multiplying a large number by a small one (in our case a^{l+1} by $\hat{v}_{mn}^{(l)}(t)$) does not arise and a relatively small number of iterations N_v is sufficient.

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