



Stationary Solutions for a Boundary Controlled Burgers' Equation

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Abstract—This paper is concerned with the stationary solutions of a one-parameter family of boundary control problems for a forced viscous Burgers' equation. We assume that the forcing term possesses a special symmetry that greatly aids in our analysis. The parameter characterizing the family enters as a scalar gain in a proportional error boundary feedback control scheme. We show that as the gain varies from zero to infinity, the stationary solutions undergo an interesting bifurcation. Namely, when the gain is zero, there are infinitely many stationary solutions, the one-dimensional subspace of all constants. When the gain is positive, the constants are no longer solutions. For small positive values of the gain, there are three distinct nonconstant stationary solutions, and for sufficiently large values of the gain there is a single, asymptotically stable equilibrium. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Burgers' equation provides a remarkable system (see, e.g., [1]) that has been studied for some time [2] and was extensively developed by Burgers [3] as a simplified fluid flow model which nonetheless exhibits some of the important aspects of turbulence. It was later derived by Lighthill [4] as a second-order approximation to the one-dimensional unsteady Navier-Stokes equation.

In a series of more recent papers, which resulted in [5], the viscous Burgers' equation has been examined within the context of boundary control leading to a study of boundary value problems very different from those usually treated in the literature. For example, the case of periodic boundary conditions can be found in [6] and a study of Burgers' equation with Dirichlet boundary conditions can be found in [7]. In [7], the authors show that for a given forcing term, there is a single, global, asymptotically stable equilibrium. The methods employed in that work are straightforward; they employ the famous Hopf-Cole transformation [8] and reduce the problem

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to the study of a standard boundary value problem for the heat equation (see also [9,10] for other interesting explicit solutions for Burgers' equation with boundary conditions). The same type of analysis for Burgers' equation for other boundary conditions is not possible for reasons that will become clear in the present paper. Namely, Neumann boundary conditions are transformed into quadratic nonlinear boundary conditions which also involve the time derivative, as well as the second-order spatial derivative of the solution on the boundary.

The papers [11,12] first announced the existence of a local attractor for each value of a certain parameter occurring in the boundary conditions. A next natural and important problem is to determine the structure of the attractor for all values of the parameter. It turns out that the analysis of this problem is not easy. For one thing, it has not been possible to obtain the existence of a global absorbing ball for the solutions and, in fact, such a ball may not even exist. Furthermore, we have not been able to construct a global Lyapunov function. In any case, the starting point for analyzing the structure of the attractor is certainly to understand the number and properties of stationary solutions. In an earlier work [13], which was in large part based on numerical considerations, the authors concluded that the structure of the stationary solutions could be nontrivial depending on the value of the gain parameter.

Following the earlier conference proceedings paper [11] and an unpublished work [12] containing the details of the results announced in [11], several authors have recently become interested in related work for a boundary controlled Burgers' equation; cf. [14,15].

In the works [16,17], the authors considered the high gain limit (i.e., the system obtained by formally passing the gain to infinity) of the boundary control systems. Following the development of nonlinear zero dynamics as discussed by Byrnes and Isidori [18], we can define the zero dynamics system and show that the closed loop trajectories converge to trajectories of the zero dynamics as the gain tends to infinity. The zero dynamics for our boundary controlled Burgers' problem is Burgers' equation with Dirichlet boundary conditions, and it is easy to show that this problem has a trivial global attractor consisting of a single point [7]. In [16], we showed that the local attractor for the closed loop system converges to the attractor of the zero dynamics, but it remained unclear whether there existed a finite gain after which the attractor would consist of a single stationary function.

The purpose of the present work is to give a rigorous mathematical justification of the fact there can be multiple stationary points contained in the global attractor. Partial numerical results concerning this possibility were first announced in [13]. The analysis in this paper is based on an extension of the classical analysis of Sturm-Liouville boundary value problems using the so-called Prüfer transformations or polar coordinates. We should comment that various partial results were obtained earlier using a variety of approaches, but none of these approaches provided the more complete picture presented here.

Our approach proceeds as follows. We first integrate the stationary Burgers' equation and then introduce the Riccati or Hopf-Cole transformation to reduce the stationary Burgers' equation to a second-order linear equation containing a spectral type parameter. As mentioned above, the boundary conditions are transformed into nonlinear boundary conditions. In the stationary case, these boundary conditions factor into products of boundary conditions which appear to be of Sturmian type. In particular, we obtain a system of the form

$$\begin{aligned} y'' - Fy &= \lambda y, \\ y'(0) - \ell_0(\lambda)y(0) &= 0, \\ y'(1) + \ell_1(\lambda)y(1) &= 0. \end{aligned}$$

At first glance, these "eigenvalue" problems appear fairly simple, but the fact that the spectral parameter λ appears in a nontrivial way in the boundary conditions leads to technical difficulties. The analysis of these strange spectral problems may well be of some independent interest.

The paper is organized as follows. First, in Section 2, we present the controlled Burgers' problem and some motivating remarks for obtaining the closed loop boundary control problem. We then introduce certain restrictions on the forcing term and describe the stationary Burgers' problem. At the end of Section 2, we state the main result of this work in Theorem 2.1. The proof of Theorem 2.1 is contained in a sequence of lemmas and remarks in Sections 3–5. In Section 3, we introduce the Hopf-Cole (or more precisely, in this development, the Riccati transformation) and describe the resulting Sturmian type spectral problems. We then reformulate the stationary problem using the classical polar coordinate transformations into four initial and boundary value problems. Section 4 contains the justification for the existence of the stationary solutions described in Theorem 2.1. Finally, in Section 5, we prove the final result of Theorem 2.1; namely, for large values of the gain parameter, the single equilibrium is asymptotically stable.

2. BURGERS' SYSTEM, ASSUMPTIONS AND MAIN RESULT

Consider the controlled viscous Burgers' system

$$\begin{aligned} w_t - \epsilon w_{xx} + w w_x &= f(x), \\ w &= w(x, t), \quad x \in (0, 1), \quad t \geq 0, \\ -w_x(0, t) &= u_0(t), \quad w_x(1, t) = u_1(t), \\ w(x, 0) &= \phi(x), \\ y_0(t) &= w(0, t), \quad y_1(t) = w(1, t), \end{aligned} \tag{2.1}$$

where $u_0(t)$, $u_1(t)$ are boundary inputs, $y_0(t)$, $y_1(t)$ are boundary outputs, and $f \in L^2(0, 1)$ is an external forcing term modeling an unknown disturbance.

Formally introducing proportional error feedback in the form

$$u_0 = -k y_0, \quad u_1 = -k y_1, \tag{2.2}$$

with feedback gain $k > 0$, we obtain the closed loop Burgers' system

$$\begin{aligned} w_t - \epsilon w_{xx} + w w_x &= f, \\ x &\in (0, 1), \quad t \geq 0, \\ -w_x(0, t) + k w(0, t) &= 0, \\ w_x(1, t) + k w(1, t) &= 0, \\ w(x, 0) &= \phi(x). \end{aligned} \tag{2.3}$$

In this paper, we are interested in forcing terms $f(x)$ possessing a certain symmetry property and an additional definiteness property. These assumptions considerably simplify the analysis of the resulting stationary problem.

ASSUMPTION 2.1. *We assume that f is an odd function about $x = 1/2$ in the interval $[0, 1]$, i.e.,*

$$f(x) = -f(1 - x), \quad \text{for } x \in [0, 1],$$

and we introduce the terminology “antisymmetric about $1/2$ ” or simply “antisymmetric” to describe such a function.

We will also assume that

$$f(x) > 0, \quad \text{for } x \in \left[0, \frac{1}{2}\right).$$

An important fallout of the antisymmetry condition is that it is preserved by solutions of (2.3).

The stationary Burgers' system associated with (2.3) is

$$\begin{aligned} -\epsilon w_{xx}(x) + w(x)w_x(x) &= f(x), \\ w_x(0) - k w(0) &= 0, \\ w_x(1) + k w(1) &= 0. \end{aligned} \tag{2.4}$$

Our main result is the following theorem.

THEOREM. *The following statements hold.*

- (a) *For any $k > 0$, there exists an antisymmetric solution of the stationary Burgers' system (2.4). This solution is the unique solution of (2.4) for sufficiently large k .*
- (b) *For sufficiently small k , in addition to the antisymmetric stationary solution, there also exist at least two nonantisymmetric stationary solutions.*
- (c) *The antisymmetric stationary solution is asymptotically stable for sufficiently large k .*

3. TRANSFORMATION OF THE PROBLEM

In this section, we will introduce several transformations to obtain a form of system (2.4) which will be used to facilitate our analysis.

Integrating the differential equation (2.4) over the interval $[0, x]$, we arrive at

$$-\epsilon w'(x) + \frac{w(x)^2}{2} = \int_0^x f(s) ds + c, \quad (3.1)$$

where c is a constant of integration. Using the boundary conditions, we see that

$$c = -\epsilon k w(0) + \frac{w(0)^2}{2} \quad (3.2)$$

is actually not arbitrary. The parameter c plays an important part in our analysis.

Equation (3.1) is a Riccati ordinary differential equation and one classical approach to solving this equation is to introduce the so-called Riccati transformation, i.e., we seek a solution in the form

$$w(x) = -2\epsilon \frac{v'(x)}{v(x)}. \quad (3.3)$$

Using (3.3), the differential equation (3.1) is transformed into

$$v''(x) - (F(x) + \lambda)v(x) = 0, \quad (3.4)$$

where

$$F(x) = \frac{1}{2\epsilon^2} \left(\int_0^x f(s) ds \right) \quad (3.5)$$

and

$$\lambda = \frac{c}{2\epsilon^2}. \quad (3.6)$$

As we mentioned in the Introduction, the boundary conditions are transformed into nonlinear boundary conditions, which, after using equations (3.4) and (3.5), can be written as

$$\begin{aligned} -cv(0)^2 + 2v'(0)^2\epsilon^2 + 2kv'(0)v(0)\epsilon^2 &= 0, \\ -cv(1)^2 + 2v'(1)^2\epsilon^2 - 2kv'(1)v(1)\epsilon^2 &= 0. \end{aligned} \quad (3.7)$$

These conditions can be factored into products of conditions that appear to be of ‘‘Sturm-Liouville’’ type

$$(v'(0) - l^+v(0)) (v'(0) - l^-v(0)) = 0, \quad (3.8)$$

$$(v'(1) + l^+v(1)) (v'(1) + l^-v(1)) = 0. \quad (3.9)$$

However, the parameters l^\pm depend on the ‘‘spectral’’ parameter λ ,

$$l^\pm = l^\pm(k, \lambda) = -\frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 + \lambda}. \quad (3.10)$$

Considering the various possibilities, we arrive at four systems that can deliver stationary solutions

$$v''(x) - (F(x) + \lambda)v(x) = 0, \quad (3.11)$$

$$\begin{aligned} v'(0) - l^+v(0) = 0, & \quad v'(1) + l^+v(1) = 0, \\ v'(0) - l^+v(0) = 0, & \quad v'(1) + l^-v(1) = 0, \\ v'(0) - l^-v(0) = 0, & \quad v'(1) + l^+v(1) = 0, \\ v'(0) - l^-v(0) = 0, & \quad v'(1) + l^-v(1) = 0. \end{aligned} \quad (3.12)$$

We are interested in nonvanishing solutions, because solutions of the stationary Burgers' equation are given by $-2\epsilon(v'(x)/v(x))$, and therefore, the existence of a zero of $v(x)$ together with uniqueness of solutions of the initial value problem would imply a blow-up solution (this will be clear from the next section).

We next introduce the classical Prüfer transformation for (3.11),(3.12), namely,

$$v(x) = r(x) \cos \phi(x), \quad (3.13)$$

$$v'(x) = r(x) \sin \phi(x). \quad (3.14)$$

Differentiating (3.13) yields

$$v'(x) = r'(x) \cos \phi(x) - r(x) \sin \phi(x) \phi'(x). \quad (3.15)$$

Now combine (3.14) and (3.15), to get

$$r(x) \sin \phi(x) = r'(x) \cos \phi(x) - r(x) \sin \phi(x) \phi'(x). \quad (3.16)$$

From (3.11), we have

$$v''(x) = (F(x) + \lambda)r(x) \cos \phi(x), \quad (3.17)$$

and differentiating (3.14) gives

$$v''(x) = r'(x) \sin \phi(x) + r(x) \cos \phi(x) \phi'(x). \quad (3.18)$$

Combining (3.17) and (3.18) gives

$$(F(x) + \lambda)r(x) \cos \phi(x) = r'(x) \sin \phi(x) + r(x) \cos \phi(x) \phi'(x). \quad (3.19)$$

Multiplying (3.16) by $\cos \phi(x)$ and adding to (3.19) times $\sin \phi(x)$ gives

$$r'(x) = (1 + F(x) + \lambda)r(x) \sin \phi(x) \cos \phi(x). \quad (3.20)$$

Equation (3.20) can be solved in terms of ϕ as

$$r(x) = r(0) \exp \left(\int_0^x (1 + F(s) + \lambda) \sin \phi(s) \cos \phi(s) ds \right). \quad (3.21)$$

Multiplying (3.19) by $\cos \phi(x)$ and subtracting (3.16) times $\sin \phi(x)$ gives

$$r(x) \phi'(x) = (F(x) + \lambda)r(x) \cos^2 \phi(x) - r(x) \sin^2 \phi(x). \quad (3.22)$$

Since we are looking for solutions $v(x)$ which have no zeros on $[0, 1]$ and also, since (3.21),(3.22) are linear in $r(x)$, we may assume, without loss of generality, that $r(x) > 0$ for $x \in [0, 1]$. Using this assumption, we can divide by $r(x)$ in (3.22) to obtain

$$\phi'(x) = (F(x) + \lambda) \cos^2 \phi(x) - \sin^2 \phi(x). \quad (3.23)$$

In what follows, equation (3.23) is the main equation used in our analysis. On one hand, it is independent of $r(x)$, and on the other hand, $r(x)$ is completely determined by $r(0)$ and $\phi(x)$ in (3.21).

Rewriting the boundary conditions (3.12), taking into account that $r(0)$ and $r(1)$ are not zero, and after some simplification using (3.23), we obtain the following four sets of boundary value problems:

$$\phi'(x) = (F(x) + \lambda) \cos^2 \phi(x) - \sin^2 \phi(x), \quad (3.24)$$

$$\phi(0) = \arctan(\ell^{s_1}), \quad (3.25)$$

$$\phi(1) = -\arctan(\ell^{s_2}), \quad (3.26)$$

where $s_1, s_2 \in \{+, -\}$.

REMARK 3.1.

1. Note that (3.24),(3.25) give initial value problems, (IVP). For these problems, elementary results from the theory of ordinary differential equations ensure that there exist unique solutions which depend continuously on k and λ .
2. If $\phi(x)$ denotes the solution to an IVP, then we will sometimes use the notation $\phi(x, k, \lambda)$ to emphasize the dependence on k and λ .
3. We will refer to the problem (3.24)–(3.26) as the boundary value problem, (BVP).
4. The problem we want to solve can now be stated as follows:
for a fixed $k > 0$, find λ so that the solution $\phi(x)$ of the IVP (3.24),(3.25) satisfies, in addition, (3.26), i.e., the BVP.
5. Because we are looking for functions $v(x)$ with no zeros on $[0, 1]$, we are interested in solutions $\phi(x)$ to the IVP, that take values in the interval $(-\pi/2, \pi/2)$.
6. We will use the notation ϕ^{s_1} to denote the solution of the IVP with $\phi(0) = \arctan(\ell^{s_1})$. For the BVPs (s_1, s_2) with $s_j \in \{+, -\}$, $j = 1, 2$, the corresponding solution will be denoted by $\phi^{s_1, s_2}(x)$.
7. In the cases $(+, +)$ and $(-, -)$, we have the following.
 - (a) It is clear from the form of the conditions in (3.25),(3.26) that, if ϕ is a solution of the BVP, then $\phi(1) = -\phi(0)$.
In fact, if $\phi(1) = -\phi(0)$ where ϕ is a solution to the BVP (3.24)–(3.26), then $-\phi(1-x)$ is also a solution, hence, by the uniqueness of solutions to the IVP, we have $\phi(x) = -\phi(1-x)$, i.e., the solution is antisymmetric. This implies that the corresponding stationary solution $w(x)$ of (2.4) is also antisymmetric.
From $\phi(1) = -\phi(0)$, it also follows that $\phi(1/2) = 0$.
8. More generally, note that if $w(x, t)$ is the solution of (2.3) for antisymmetric initial data and forcing term, then $w(x, t) = -w(1-x, t)$ for all $t \geq 0$, i.e., w is antisymmetric.
9. In the $(-, +)$, $(+, -)$ cases, a solution $\phi(x)$ of the BVP (3.24)–(3.26) and the corresponding solution $w(x)$ of the steady-state Burgers' equation are nonantisymmetric. Nevertheless, there is a useful symmetry. If ϕ is a solution of the BVP for $(+, -)$, then $-\phi(1-x)$ is a solution of the BVP $(-, +)$, and vice versa.
10. We are only interested in real valued solutions.

From the form of ℓ^s , we see that this imposes a restriction on the possible values for λ , namely,

$$-\left(\frac{k}{2}\right)^2 \leq \lambda < \infty.$$

The proof of Theorem 2.1 is obtained through a sequence of lemmas contained in Sections 4 and 5.

4. EXISTENCE OF SOLUTIONS

This section contains the proof of Statements (a) and (b) of Theorem 2.1. We obtain these results from Lemmas 4.1–4.6 and 4.8 below. Lemmas 4.7 and 4.8 prove Statement (b).

LEMMA 4.1. *Let $\phi_1(x)$, $\phi_2(x)$ be two solutions of the IVP corresponding to λ_1 and λ_2 , respectively.*

If $\lambda_1 \neq \lambda_2$ and $\phi_1(x), \phi_2(x) \in (-\pi/2, \pi/2)$, then the graphs of $\phi_1(x)$ and $\phi_2(x)$ can have only finitely many intersections on $[0, 1]$.

PROOF. Suppose there are infinitely many intersections, i.e., there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset [0, 1]$ such that $\phi_1(x_n) = \phi_2(x_n)$ for $n = 0, 1, \dots$. Then there is an $\tilde{x} \in [0, 1]$ and a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ such that $x_{n_k} \rightarrow \tilde{x}$ as $k \rightarrow \infty$. Let us once again denote this subsequence by x_n . Since $\phi_1(x_n) - \phi_2(x_n) = 0$ for all n , and $\phi_1(x) - \phi_2(x)$ is continuous, we have $\phi_1(\tilde{x}) - \phi_2(\tilde{x}) = 0$.

By Rolle's Theorem, there also exists a sequence $\{\xi_n\}_{n=0}^{\infty}$ such that $\xi_n \in [x_n, x_{n-1}]$ and $\phi_1'(\xi_n) - \phi_2'(\xi_n) = 0$. Once again, we have $\phi_1'(\xi_n) - \phi_2'(\xi_n) \rightarrow \phi_1'(\tilde{x}) - \phi_2'(\tilde{x})$ as $n \rightarrow \infty$, and therefore, $\phi_1'(\tilde{x}) = \phi_2'(\tilde{x})$.

Using (3.24) with $\phi_1(\tilde{x}) = \phi_2(\tilde{x})$, $\phi_1'(\tilde{x}) = \phi_2'(\tilde{x})$, we obtain $(\lambda_1 - \lambda_2) \cos^2(\phi_1(\tilde{x})) = 0$ which implies that $\lambda_1 = \lambda_2$ or $\phi_1(\tilde{x}) = \pm(\pi/2)$, which is a contradiction since $\lambda_1 \neq \lambda_2$ and $\phi_1(\tilde{x}) \neq \pm(\pi/2)$. Therefore, there are only finitely many intersections. ■

4.1. The Cases (+, +) and (−, −)

We begin this section with a remark containing several pieces of useful information.

REMARK 4.1. Recall our notation in Remark 3.1, part (6); namely, ϕ^{\pm} denotes the solution to the IVP, i.e., it satisfies equation (3.25) and the initial condition (3.26),

$$\phi^{\pm}(0, k, \lambda) = \arctan(\ell^{\pm}) = \arctan\left(-\frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 + \lambda}\right).$$

This implies that for a fixed k , as λ varies over the interval $-(k/2)^2 \leq \lambda < \infty$,

1. the numbers $\phi^+(0, k, \lambda)$ cover the interval $[-\arctan(k/2), \pi/2)$;
2. the numbers $\phi^-(0, k, \lambda)$ cover the interval $(-\pi/2, -\arctan(k/2)]$;
3. in particular, we note that

$$\phi^-\left(0, k, -\left(\frac{k}{2}\right)^2\right) = \phi^+\left(0, k, -\left(\frac{k}{2}\right)^2\right) = -\arctan\left(\frac{k}{2}\right);$$

4. $\phi^+(0, k, 0) = 0$;
5. $\phi^+(0, k, \lambda) > 0$ for $\lambda > 0$ (see Figure 1).

LEMMA 4.2. *For any $k \geq 0$ and $\lambda \geq 0$, we have $\phi^+(1) \equiv \phi^+(1, k, \lambda) > 0$.*

PROOF. We know that $\phi^+(0) \geq 0$, and $\phi^+(0) = 0$ if and only if $\lambda = 0$. In this latter case,

$$\phi^{+'}(0) = F(0) \cos^2 \phi^+(0) - \sin^2 \phi^+(0) = 0$$

and

$$\phi^{+''}(0) = F'(0) = f(0) > 0.$$

Therefore, in either of the cases $\phi^+(x)$ is positive in a neighborhood of the origin for $\lambda \geq 0$.

Suppose that $\phi^+(1) < 0$.

This implies that the solution has to cross the x -axis in order to reach $\phi^+(1) < 0$. Let $x_0 \in (0, 1)$ be the first zero crossing. (An argument similar to the argument used in Lemma 4.1 shows that there is a first zero crossing.)

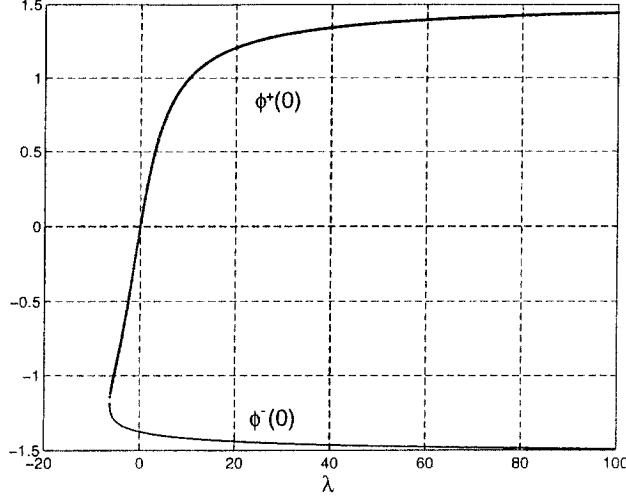


Figure 1. $\phi^+(0, k, \lambda)$, $\phi^-(0, k, \lambda)$ for $k = 5$.

Since $\phi^+(x) > 0$ for $x \in (0, x_0)$, we see that $\phi^{+'}(x_0) \leq 0$. But from (3.24), we have $\phi^{+'}(x_0) = (F(x_0) + \lambda) \geq F(x_0) > 0$, since $F(x) > 0$ on $(0, 1)$. This leads to a contradiction, and therefore, $\phi^+(1)$ cannot be negative.

Suppose that $\phi^+(1) = 0$.

In this case, $\phi^+(0) = \phi^+(1)$, i.e., $\phi^+(x)$ must be antisymmetric, which implies that $\phi^+(1/2) = 0$. From here, we get a contradiction the same way as before since there must be a first zero in $0 < x_0 \leq 1/2$.

Thus, we conclude that $\phi^+(1) > 0$. ■

LEMMA 4.3. *In the following cases, we have $\phi^-(1) < \phi^-(0) < 0$:*

1. $k \geq \sqrt{2F(1/2)}$ and $\lambda \geq -(k/2)^2$;
2. $k < \sqrt{2F(1/2)}$ and $\lambda > (1/k^2)(F(1/2) - k^2/2)^2 - (k/2)^2$.

PROOF. From equation (3.24),

$$\begin{aligned}
 \phi^{-'}(0) &= \lambda \cos^2 \phi^-(0) - \sin^2 \phi^-(0) \\
 &= \lambda \cos^2 (\arctan (l^-)) - \sin^2 (\arctan (l^-)) \\
 &= \frac{\lambda}{1 + (l^-)^2} - \frac{(l^-)^2}{1 + (l^-)^2} \\
 &= \frac{-k^2/2 - k\sqrt{(k/2)^2 + \lambda}}{1 + (l^-)^2} \\
 &< 0.
 \end{aligned} \tag{4.1}$$

Hence, $\phi^-(x) < \phi^-(0)$ in a neighborhood of the origin.

Suppose that $\phi^-(1) \geq \phi^-(0)$. Then there exists an $\tilde{x} \in (0, 1)$ such that $\phi^{-'}(\tilde{x}) = 0$ and $\phi^-(\tilde{x}) < \phi^-(0) < 0$.

But from (3.24),

$$\begin{aligned}
 \phi^{-'}(\tilde{x}) &= (F(\tilde{x}) + \lambda) \cos^2 \phi^-(\tilde{x}) - \sin^2 \phi^-(\tilde{x}) \\
 &< \left(F\left(\frac{1}{2}\right) + \lambda \right) \cos^2 \phi^-(0) - \sin^2 \phi^-(0) \\
 &= \frac{F(1/2) - k^2/2 - k\sqrt{(k/2)^2 + \lambda}}{1 + (l^-)^2},
 \end{aligned}$$

which is negative under the assumptions on k and λ . This contradiction proves the lemma. ■

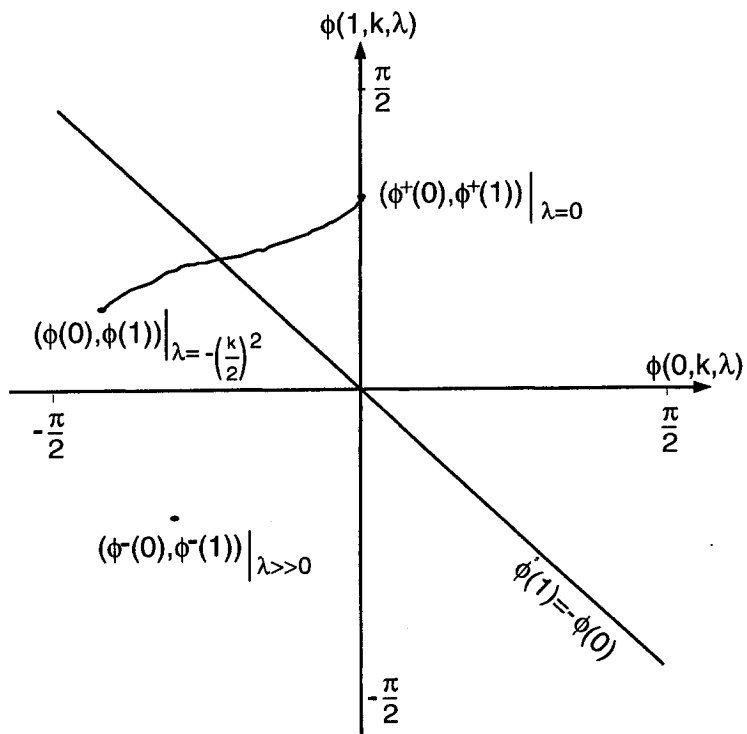
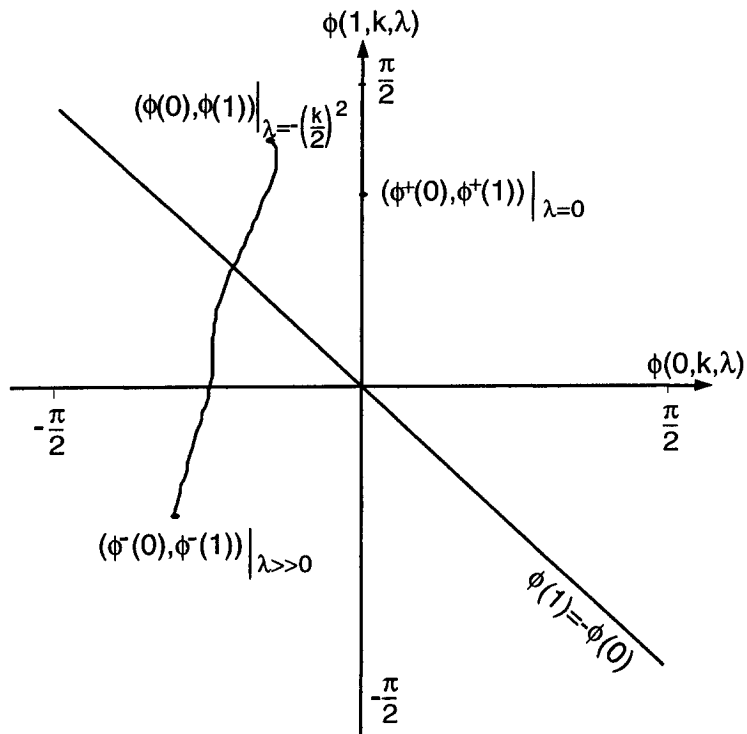


Figure 2.

LEMMA 4.4. $\phi^+(1) > -\phi^+(0)$ for small positive values of k and all $\lambda \geq -(k/2)^2$.

PROOF. $\phi^+(1) > 0 = -\phi^+(0)$ for $k = 0$ and $\lambda = 0$ by Lemma 4.2. It then follows from continuous dependence that for small positive k and for all $0 > \lambda \geq -(k/2)^2$, we still have $\phi^+(1) \geq -\phi^+(0)$.

In case of $\lambda \geq 0$, the claim holds for all $k > 0$ again by Lemma 4.2. ■

Figure 2 summarizes the previous results of this section, describing the relationship between $\phi(0)$ and $\phi(1)$ for various λ and k . To show how Figure 2 can be used to demonstrate the existence of stationary solutions, we now collect several points that were made in Lemmas 4.2–4.4.

- a. The point $(\phi^+(0), \phi^+(1))|_{\lambda=0}$ lies on the positive $\phi(1)$ axis according to Point 4 of Remark 4.1 and Lemma 4.2.
- b. The point $(\phi^-(0), \phi^-(1))|_{\lambda \gg 0}$ lies in the third quadrant according to Lemma 4.3.
- c. For $\lambda = -(k/2)^2$, we have $\phi^+(x) = \phi^-(x)$, and $(\phi(0), \phi(1))|_{\lambda=-(k/2)^2}$ is somewhere in the second or third quadrant according to Point 3 of Remark 4.1.
- d. Since the solution to the IVP both in the (+) and in the (–) case depends continuously on λ , we have that the point $(\phi(0), \phi(1))$ moves continuously in the $(\phi(0), \phi(1))$ -plane starting at the point $(\phi(0), \phi(1))|_{\lambda=-(k/2)^2}$ as λ increases from $\lambda = -(k/2)^2$. Its graph crosses the point $(\phi^-(0), \phi^-(1))|_{\lambda \gg 0}$ in the (–) case, and the point $(\phi^+(0), \phi^+(1))|_{\lambda=0}$ in the (+) case.

Considering the various cases based on Figure 2, we can now state the main lemma of this section.

LEMMA 4.5. *The following statements hold.*

1. *There always exists at least one antisymmetric solution.*
2. *For large k , there is no antisymmetric solution corresponding to the (–, –) case, and thus, any stationary solution for large k must come from the (+, +) case.*
3. *For small k , antisymmetric solutions must come from the (–, –) case since for small k , there are no solutions in the (+, +) case.*
4. *There exists a k for which $\lambda = -(k/2)^2$, and here, the two cases coincide.*

PROOF.

1. In order to have a solution for the BVP for a given $k > 0$, we must prove the existence of a λ for which the solution $\phi(x, k, \lambda)$ of the corresponding IVP (3.24),(3.25) satisfies, in addition, $\phi(1) = -\phi(0)$. This last condition means that the point $(\phi(0), \phi(1))$ lies on the line $\phi(1) = -\phi(0)$ in Figure 2. There are three possible cases.
 - a. If $(\phi(0), \phi(1))|_{\lambda=-(k/2)^2}$ is on the line $\phi(1) = -\phi(0)$, then $\phi(x)$ is the antisymmetric solution of the BVP.
 - b. If $(\phi(0), \phi(1))|_{\lambda=-(k/2)^2}$ is under the line $\phi(1) = -\phi(0)$, then the graph of $(\phi^+(0), \phi^+(1))$ crosses the line $\phi(1) = -\phi(0)$ at some $-(k/2)^2 < \lambda < 0$, so that there is an antisymmetric solution coming from the (+, +) case.
 - c. If $(\phi(0), \phi(1))|_{\lambda=-(k/2)^2}$ is above the line $\phi(1) = -\phi(0)$, then the graph of $(\phi^-(0), \phi^-(1))$ crosses the line $\phi(1) = -\phi(0)$ at some $-(k/2)^2 < \lambda$, so that there is an antisymmetric solution coming from the (–, –) case.

Hence, there always exists at least one antisymmetric solution.

2. $(\phi^-(0), \phi^-(1))|_{\lambda}$ is in the third quadrant for all the possible values of λ in case of large k according to Point 2 of Remark 4.1 and Point 1 of Lemma 4.3. Thus, the antisymmetric solution must correspond to the (+, +) case.
3. The points $(\phi^-(0), \phi^-(1))|_{\lambda}$ lie above the line $\phi(1) = -\phi(0)$ for all the possible values of λ in case of small positive values of k according to Lemma 4.4. That means that there are no solutions corresponding to the (+, +) case for small k , so that solutions must come from the (–, –) case.
4. It follows from the previous two points and from the continuous dependence of solutions on k , that there exists a k (not too small and not too large) for which the point $(\phi(0), \phi(1))|_{\lambda=-(k/2)^2}$ lies on the line $\phi(1) = -\phi(0)$, so that the two cases coincide.

With this the proof is complete. ■

LEMMA 4.6. *The antisymmetric solution is unique for large k .*

PROOF. Suppose that for a $k > 0$, there exist $\lambda_1 < \lambda_2$ with corresponding solutions $\phi_1^{+,+}(x)$, $\phi_2^{+,+}(x)$ for the BVP. From (3.25), we must have $\phi_1^{+,+}(0) < \phi_2^{+,+}(0)$, and hence,

$$\phi_1^{+,+}(1) = -\phi_1^{+,+}(0) > -\phi_2^{+,+}(0) = \phi_2^{+,+}(1).$$

Both functions are continuous, so there exists $x \in (0, 1)$ such that $\phi_1^{+,+}(x) = \phi_2^{+,+}(x)$. According to Lemma 4.1, there are finitely many such points, and hence, there is a first one which we denote by x_0 . We have $\phi_1^{+,+}'(x_0) \geq \phi_2^{+,+}'(x_0)$ since $\phi_1^{+,+}(x_0) = \phi_2^{+,+}(x_0)$ and $\phi_1^{+,+}(x) < \phi_2^{+,+}(x)$ for $x \in [0, x_0)$. But from (3.24), using the notation

$$\phi(x_0) \equiv \phi_1^{+,+}(x_0) = \phi_2^{+,+}(x_0),$$

with the assumption that $\lambda_1 < \lambda_2$, we have

$$\begin{aligned} \phi_1^{+,+}'(x_0) &= (F(x_0) + \lambda_1) \cos^2 \phi(x_0) - \sin^2 \phi(x_0) \\ &< (F(x_0) + \lambda_2) \cos^2 \phi(x_0) - \sin^2 \phi(x_0) \\ &= \phi_2^{+,+}'(x_0), \end{aligned}$$

which leads to a contradiction. This contradiction proves the uniqueness of the solution. \blacksquare

4.2. The Cases $(-, +)$ and $(+, -)$

LEMMA 4.7. *$\phi(x)$ is a solution of the BVP in the $(-, +)$ case if and only if*

$$\tilde{\phi}(x) = -\phi(1-x)$$

is a solution to the BVP in the $(+, -)$ case.

PROOF. Since $\tilde{\phi}(x) = -\phi(1-x)$ implies $\phi(x) = -\tilde{\phi}(1-x)$, it is enough to prove only one direction. Thus, we assume that $\phi(x)$ is a solution of the BVP in the $(-, +)$ case. Then using the symmetry property of F , \cos^2 , and \sin^2 , we obtain

$$\begin{aligned} \tilde{\phi}'(x) &= (-\phi(1-x))' = \phi'(1-x) \\ &= (F(1-x) + \lambda) \cos^2 \phi(1-x) - \sin^2 \phi(1-x) \\ &= (F(x) + \lambda) \cos^2 (-\phi(1-x)) - \sin^2 (-\phi(1-x)) \\ &= (F(x) + \lambda) \cos^2 \tilde{\phi}(x) - \sin^2 \tilde{\phi}(x), \end{aligned} \tag{4.2}$$

so that $\tilde{\phi}$ satisfies (3.23). Also,

$$\tilde{\phi}(0) = -\phi(1) = -\arctan(-l^-) = \arctan(l^-) \tag{4.3}$$

and

$$\tilde{\phi}(1) = -\phi(0) = -\arctan(l^+), \tag{4.4}$$

which shows that $\tilde{\phi}$ satisfies the BVP for the $(+, -)$ case. \blacksquare

According to the previous lemma, it is enough to consider the $(-, +)$ case.

LEMMA 4.8. *For small positive values of k , there exists a λ such that the BVP with $(-, +)$ has a nonantisymmetric solution. For large k , there is no nonantisymmetric solution to the BVP.*

PROOF. According to Lemma 4.4 and Remark 4.1.3, we have

$$\begin{aligned} \phi^- \left(1, k, - \left(\frac{k}{2} \right)^2 \right) &= \phi^+ \left(1, k, - \left(\frac{k}{2} \right)^2 \right) \\ &> -\phi^+ \left(0, k, - \left(\frac{k}{2} \right)^2 \right) \\ &= \arctan \left(\frac{k}{2} \right) \\ &> \arctan \left(-\frac{k}{2} \right) = \arctan(\ell^+) \end{aligned}$$

for small positive values of k .

On the other hand, according to Lemma 4.3, $\phi^-(1, k, \lambda) < \phi^-(0, k, \lambda) < \arctan(\ell^+)$ at least for large values of λ . It follows from the continuous dependence on the parameter λ of solutions to the IVP, that there exists a λ for which the solution of the IVP satisfies $\phi^-(1) = \arctan(\ell^+)$, i.e., it satisfies the BVP.

The nonexistence of nonantisymmetric solutions for large k , in particular for $k > \sqrt{2F(1/2)}$, follows from Lemma 4.3. ■

5. STABILITY OF ANTISYMMETRIC STATIONARY SOLUTIONS

In this section, we prove Statement (c) of Theorem 2.1. We recall (see, e.g., [6]) that to prove the stability of a stationary solution to a boundary value problem for the Burgers' equation it is sufficient to consider the linearization of the problem about this solution. The stability problem is then reduced to the study of the location of the spectrum of the corresponding linear operator. More precisely, one has to show that the spectrum, of the linearization about the equilibrium of the spatial operator, is contained in the open left half-plane.

LEMMA 5.1. *For $k > 0$, let $a_k(x)$ denote the antisymmetric solution of the steady state Burgers' system (2.3) whose existence is guaranteed by Theorem 2.1. Then*

$$0 < a_k(0) \leq \frac{1}{k} F \left(\frac{1}{2} \right).$$

PROOF. Using notations (3.5) and (3.2), we can write equation (3.1) and the corresponding boundary conditions in the form

$$\begin{aligned} w'(x) &= \frac{w(x)^2}{2\epsilon} - 2\epsilon F(x) + w(0) \left(k - \frac{w(0)}{2\epsilon} \right) \\ &= \frac{w(x)^2}{2\epsilon} - \frac{w(0)^2}{2\epsilon} - 2\epsilon F(x) + w(0)k, \\ w'(0) - kw(0) &= 0, \\ w'(1) + kw(1) &= 0. \end{aligned} \tag{5.1}$$

Suppose that $a_k(0) > F(1/2)/k$. In this case, using (5.1), we observe that $a'_k(x) > 0$ for all $x \in (0, 1)$. Indeed, from the boundary conditions in (5.1), $a'_k(0) = ka_k(0) > 0$, and we cannot have $a_k(x) = 0$ on $(0, 1)$, because if $\tilde{x} \in (0, 1]$ is the first such point, then $a_k(\tilde{x}) > a(0)$ implies

$$a'_k(\tilde{x}) \geq \frac{a_k^2(\tilde{x}) - a_k^2(0)}{2\epsilon} - 2\epsilon F(\tilde{x}) + ka_k(0) > \frac{a_k^2(\tilde{x}) - a_k^2(0)}{2\epsilon} > 0.$$

Now $a'_k(x) > 0$ for all $x \in (0, 1)$ implies $a_k(1) > a_k(0) > 0$, which is impossible for an antisymmetric solution, and hence, we must have $a_k(0) \leq (1/k)F(1/2)$.

Suppose that $a_k(0) < 0$. Then $a_k(x) < 0$ for all $x \in [0, 1]$, because the existence of a zero would imply that at the first such zero $\tilde{x} \in (0, 1]$, the slope of the tangent line would be negative,

$$a'_k(\tilde{x}) = -2\epsilon F(\tilde{x}) + a_k(0) \left(k - \frac{a_k(0)}{2\epsilon} \right) < 0,$$

which is impossible.

The proof that $a_k(0) \neq 0$ is similar to the previous case. Suppose that $a'_k(0) = 0$; then

$$a''_k(0) = \frac{a_k(0)a'_k(0)}{\epsilon} - \frac{1}{\epsilon}f(0) = -\frac{1}{\epsilon}f(0) < 0$$

would imply that $a_k(0) < 0$ in a neighborhood of zero. Therefore, the antisymmetric solution would have to have another zero $\tilde{x} \in (0, 1/2]$ with nonnegative derivative at that point. But this is not possible since

$$a'_k(\tilde{x}) = -2\epsilon F(\tilde{x}) < 0$$

because of our assumptions on f . These contradictions prove the lemma. \blacksquare

We now linearize (2.3) about the steady-state solution $a(x)$, i.e., $w(x, t) = a(x) + v(x, t)$ where $v(x, t) \ll 1$. Then from (2.3) the linear equation for v is

$$v_t = \epsilon v_{xx} - (av)_x, \quad (5.2)$$

$$v_x(0, t) = kv(0, t), \quad (5.3)$$

$$v_x(1, t) = -kv(1, t).$$

We now show that the spectrum of the linear operator $\epsilon v_{xx} - (av)_x$ with domain defined by the boundary conditions (5.3) is real and negative for sufficiently large k .

LEMMA 5.2. *Suppose that ψ, λ is an eigenpair for the eigenvalue problem*

$$\begin{aligned} \lambda\psi &= \epsilon\psi_{xx} - (a\psi)_x, \\ \psi_x(0) &= k\psi(0), \\ \psi_x(1) &= -k\psi(1). \end{aligned} \quad (5.4)$$

Then the eigenvalues are real and negative for large k . In particular, this is true for

$$k > \sqrt{\frac{1}{\epsilon}F\left(\frac{1}{2}\right)}.$$

PROOF. It is easy to see that the eigenvalue problem is self-adjoint, and therefore, the eigenvalues are real. Since equation (5.4) is linear, and since $\psi(0) = 0$ would imply that ψ is identically zero (by uniqueness of solutions to the initial value problem), we can without the loss of generality, assume that $\psi(0) > 0$. Consider the following two cases.

CASE 1. $\psi(x)$ has a zero on $(0, 1)$.

Let \tilde{x} denote the first of these zeros. Integrating (5.4) on the interval $[0, \tilde{x}]$, we obtain

$$\begin{aligned} \lambda \int_0^{\tilde{x}} \psi(s) ds &= \epsilon\psi_x(\tilde{x}) - \epsilon\psi_x(0) - a(\tilde{x})\psi(\tilde{x}) + a(0)\psi(0) \\ &= a(0)\psi(0) + \epsilon\psi_x(\tilde{x}) - \epsilon k\psi(0) \\ &\leq (a(0) - \epsilon k)\psi(0) \\ &\leq \left(\frac{1}{k}F\left(\frac{1}{2}\right) - \epsilon k \right) \psi(0) \\ &< 0, \end{aligned} \quad (5.5)$$

for $k > \sqrt{(1/\epsilon)F(1/2)}$.

CASE 2. $\psi(x) > 0$ on $(0, 1)$.

Integrating (5.4) on the interval $[0, 1]$, we obtain

$$\begin{aligned}
 \lambda \int_0^1 \psi(s) ds &= \epsilon \psi_x(1) - \epsilon \psi_x(0) - a(1)\psi(1) + a(0)\psi(0) \\
 &= -\epsilon k \psi(1) - \epsilon k \psi(0) + a(0)\psi(0) - a(1)\psi(1) \\
 &= \psi(1)(a(0) - \epsilon k) + \psi(0)(a(0) - \epsilon k) \\
 &= (\psi(1) + \psi(0))(a(0) - \epsilon k) \\
 &\leq (\psi(1) + \psi(0)) \left(\frac{1}{k} F\left(\frac{1}{2}\right) - \epsilon k \right) \\
 &< 0,
 \end{aligned} \tag{5.6}$$

again, for $k > \sqrt{(1/\epsilon) F(1/2)}$. ■

6. NUMERICAL SIMULATIONS

In this section, we present a numerical example with the antisymmetric forcing term $f(x) = 6(2x - 1)^5 - 24(2x - 1)$ and with $\epsilon = 1$. For this example, the bifurcation of the single stable antisymmetric stationary solution occurs at a point $1 < k^* < 2$. For $k > k^*$, we have a single stationary solution, and for $k < k^*$, there are three stationary solutions. Note that the value $\lambda = \lambda(k)$ corresponding to the nonantisymmetric solutions tends to infinity as k goes to zero. As a result, the corresponding nonantisymmetric solutions diverge to plus and minus infinity, respectively, as k tends to zero.

Table 1 describes the values of the gain parameter k and the bifurcation parameters λ for this example.

In Figure 3, we have plotted the bifurcation diagram generated using the bifurcation tracing software AUTO [19]. In Figure 4, we have plotted the three stationary solutions for two different values of k .

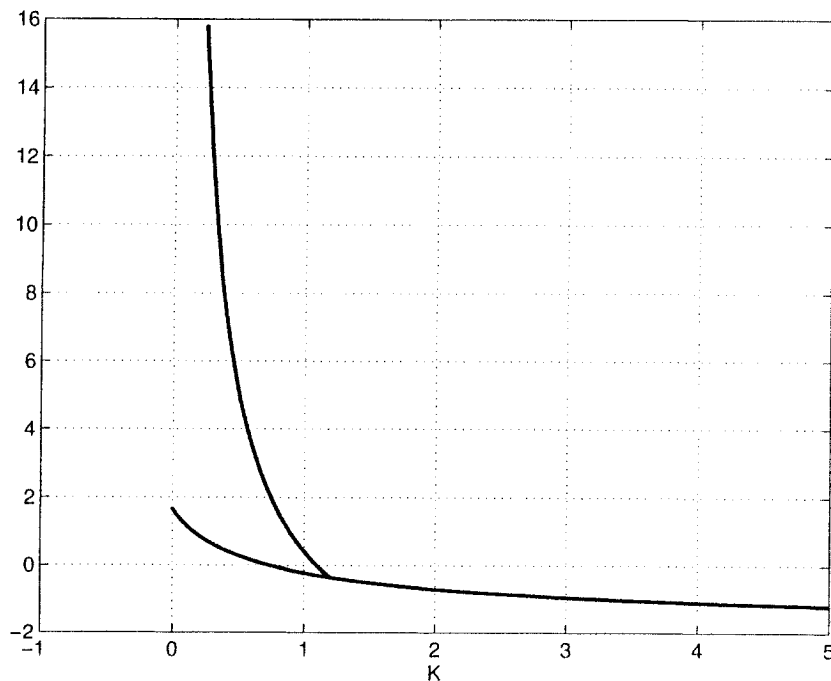
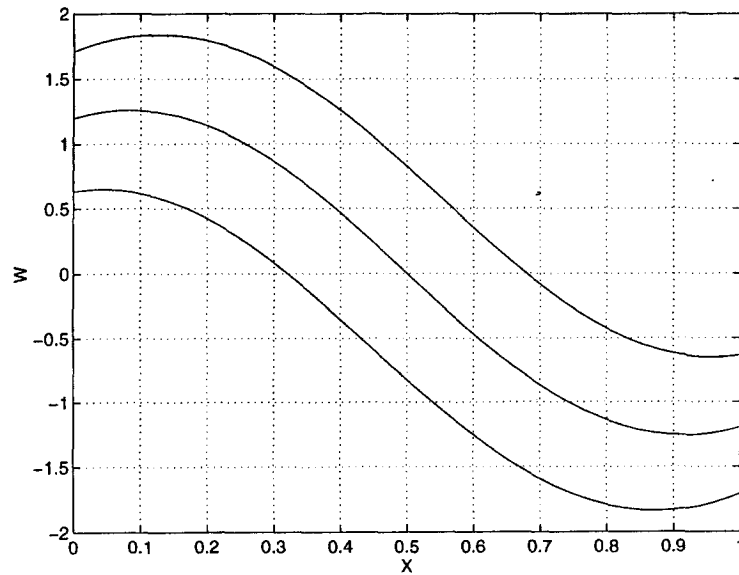


Figure 3. Bifurcation diagram for $f(x) = 6(2x - 1)^3 - 24(2x - 1)$.

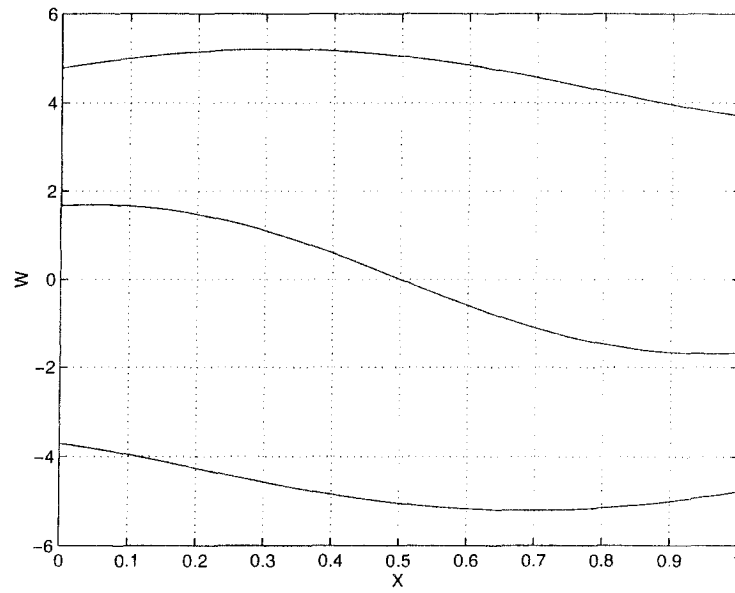
Table 1.

k	λ (as)	λ (non-as)
2.000727e + 01	-1.595890e + 00	
1.001289e + 01	-1.448489e + 00	
5.040901e + 00	-1.208515e + 00	
4.061890e + 00	-1.110018e + 00	
3.009193e + 00	-9.543289e - 01	
2.020207e + 00	-7.156459e - 01	
1.614302e + 00	-5.674827e - 01	
1.390930e + 00	-4.646846e - 01	
1.197562e + 00	-3.585385e - 01	
1.190939e + 00	-3.545605e - 01	-3.37126e - 01
1.177922e + 00	-3.466681e - 01	-2.932292e - 01
1.164994e + 00	-3.387317e - 01	-2.492107e - 01
1.152154e + 00	-3.307513e - 01	-2.062764e - 01
1.145768e + 00	-3.267448e - 01	-1.83437e - 01
1.133133e + 00	-3.187432e - 01	-1.333624e - 01
1.011015e + 00	-2.359220e - 01	3.637906e - 01
7.942373e - 01	-5.859526e - 02	1.612288e + 00
6.128592e - 01	1.318762e - 01	3.340750e + 00
3.992163e - 01	4.361554e - 01	7.352881e + 00
1.993357e - 01	8.657715e - 01	1.890458e + 01
1.104106e - 01	1.144008e + 00	3.747783e + 01
8.078343e - 02	1.256964e + 00	5.271300e + 01
6.707622e - 02	1.313764e + 00	6.421701e + 01
5.405753e - 02	1.370769e + 00	8.060602e + 01



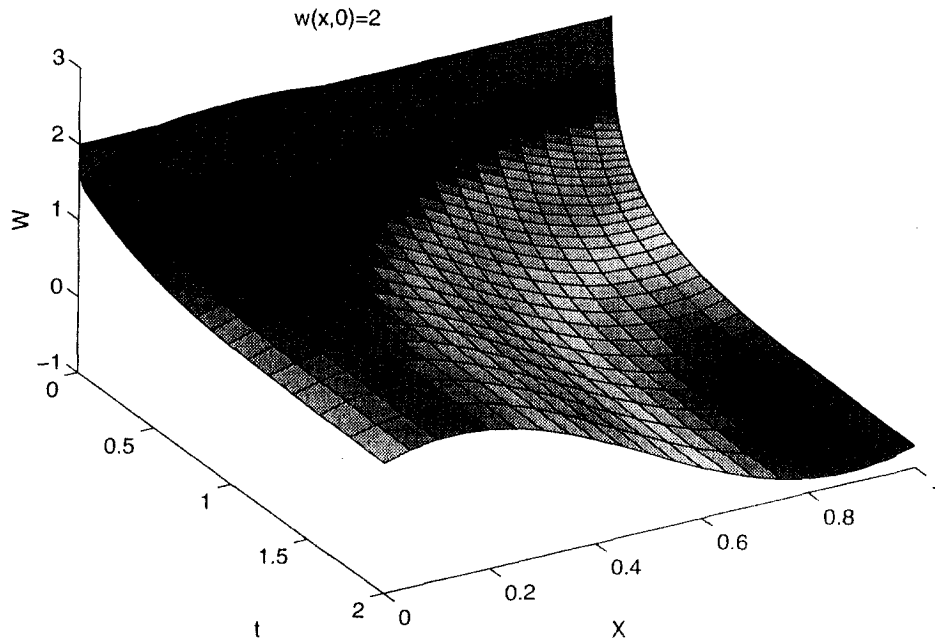
(a)

Figure 4. Multiple stationary solutions for $k < k^*$.



(b)

Figure 4. (cont.)

Figure 5. Trajectories converge to antisymmetric stationary solution, $k = 3$.

To demonstrate the stability of the antisymmetric stationary solution for large values of k , we have chosen initial data $\phi(x) = 2$, $k = 3$ and Figure 5 contains a plot of the solution surface generated numerically using the nonlinear solver XTC for t from 0 to 2. In Figure 6, we demonstrate the stability of the nonantisymmetric stationary solution for small values of k . Once again we have chosen the initial condition $\phi(x) = 2$ and, in this case, we have taken a value $k = 0.5$ which is less than the critical value k^* . Note that the trajectories converge to the positive nonantisymmetric stationary solution.

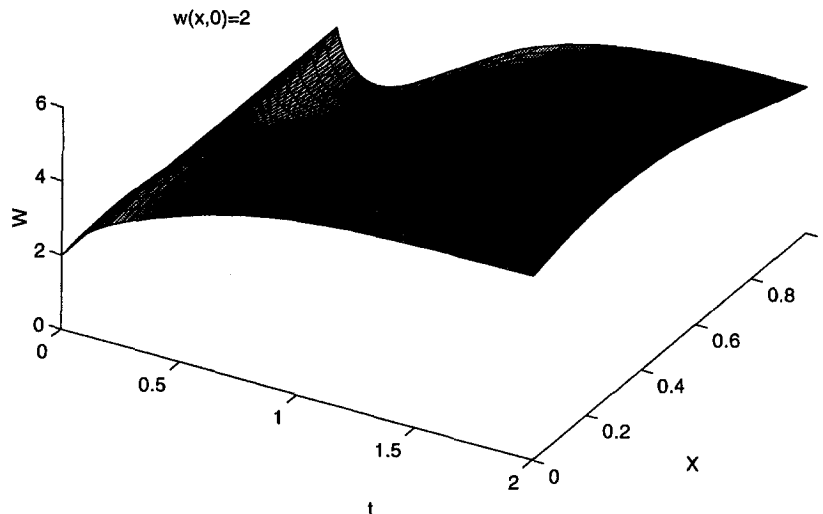


Figure 6. Trajectories converge to nonantisymmetric stationary solution, $k = 0.5$.

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