Asymptotic Distribution of Eigenvalues for Damped String Equation: Numerical Approach

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Abstract

In the present paper, we consider a one–parameter family of the nonselfadjoint operators, which are the dynamics generators for systems governed by the wave equations containing dissipative terms. The equations contain viscous damping terms and are equipped with the boundary conditions involving an arbitrary complex parameter. In the current engineering literature, this type of boundary conditions is used to model the action of smart materials (self–sensing/self–straining actuators). In the previous research of the first author, the aforementioned dynamics generators have been studied analytically and precise asymptotic formulas for the eigenvalues have been derived (the asymptotic when the number of the eigenvalues tends to infinity). *The goal of the present paper is to demonstrate that the analytic formulas are not only important theoretically, but also extremely efficient practically.* Namely, we show that the leading terms in the asymptotic formulas approximate the actual eigenvalues with excellent accuracy. To justify the results, we use two methods, i.e., the Newton method and the Tchebychev method. First, Newton's method is applied to the characteristic equation using asymptotic formulas as initial guesses to find the eigenvalues. The convergence of Newton's method is improved by modifying the asymptotic formula. Secondly, we use Tchebychev discretization to circumvent the nonlinear characteristic equation and to obtain a finite dimensional generalized eigenvalue problem that approximates the infinite dimensional one. Finally, to solve the generalized eigenvalue problem, we use the QT algorithm.

1 Introduction.

In this paper we present numerical results related to a one–parameter family of nonselfadjoint operators and nonselfadjoint quadratic operator pencils associated to those operators. The aforementioned operators are the dynamics generators governing the vibrations of a damped string with spatially inhomogeneous parameters. The present paper is a continuation of the analytical study of the model initiated in our works (Shubov $1998_1, 1997_{1,2}, 1996_1$). In that series of works, we have derived explicit asymptotic formulas for the eigenvalues of the governing operator, asymptotic when the number of an eigenvalue tends to infinity. (We present below our analytical formulas.) The purpose of the present work is to verify the practical efficiency of those formulas, i.e., we would like to obtain the answer to the following question: *If one omits the remainder terms in the asymptotic formulas for the eigenfrequencies (see formulas (24) and (25) below), then what will be the difference between the numerical value of an eigenvalue obtained by using some numerical approach and the numerical approximation for the same eigenvalue obtained from the leading term of the asymptotic?* In other words, can one identify the number of an eigenvalue such that beginning from that number, all remaining eigenvalues can be calculated within prescribed accuracy by using the leading asymptotical term only? Evidently, in practice only the first dozen of the modes and the mode shapes corresponding to the lowest frequencies play important roles. That is why it is crucially important to find efficient and accurate formulas for the lower order modes (eigenfrequencies).

At this moment we would like to emphasize that even though the present paper is a continuation of the previous research, it is self–contained and can be viewed as an independent piece of work. In

order to keep the paper self–contained, we will provide all necessary definitions and formulations of the results from our previous papers, which are necessary for understanding of the current work.

In addition to the asymptotic approximations for the eigenfrequencies, the author of the works (Shubov $1996_2, 1999$) has derived the asymptotic approximations for the eigenfunctions (mode shapes). Moreover, it has been shown that the mode shapes form *an unconditional basis* in the state space (the so–called Riesz basis). Since we consider systems with the energy dissipation, there is no way to have an orthogonal basis of the eigenfunctions; however the basis of the mode shapes, which occurs in our problem, is just a mild modification of an orthogonal basis and it is as convenient for applications as an orthogonal one. Furthermore, in our papers (Shubov and Martin 1999, Shubov 1998_2, Shubov et.al. 1997) we have solved analytically different boundary and distributed controllability problems via the spectral decomposition method. We have constructed explicitly the control laws, which could bring the vibrating systems to rest in a prescribed time interval. It is our goal in the forthcoming works to answer similar questions concerning practical efficiency of our asymptotic formulas for the control laws. *Under what conditions do the leading terms of asymptotical formulas for control laws provide practically acceptable approximations?* The latter problem is technically much more difficult.

As is well known, the key analytical result for solving different control and stabilization problems via the spectral decomposition method is *the Riesz basis property of the generalized eigenfunctions* of the appropriate dynamics generator. To prove the Riesz basis property of the generalized eigenfunctions means to show that there exists a bounded and boundedly invertible operator, which transforms the set of the generalized eigenfunctions into an orthogonal basis. Such an operator (denoted by $\mathcal{B}$) is called an *orthogonalizer*. In many applications, the aforementioned orthogonalizer can be represented in the form

$$\mathcal{B} = I + V, \qquad V \in \mathfrak{S}_2, \tag{1.1}$$

where V is a compact operator from the Hilbert–Schmidt class. If we denote the set of the generalized eigenfunctions of the dynamics generator by $\{\Phi_n\}_{n\in\mathbb{Z}}$ and an orthogonal basis by $\{\Phi_n^\circ\}_{n\in\mathbb{Z}}$, then (1.1) in fact means

$$\Phi_n - \Phi_n^\circ = V\Phi_n^\circ, \qquad n \in \mathbb{Z}, \quad V \in \mathfrak{S}_2. \tag{1.2}$$

In that case, we say that $\{\Phi_n\}_{n\in\mathbb{Z}}$ is the *Bari basis*, i.e., it is quadratically close to some orthogonal basis. Indeed, from (1.2), it immediately follows that

$$\sum_{n\in\mathbb{Z}} \|\Phi_n - \Phi_n^\circ\|_{\mathcal{H}}^2 < \infty. \tag{1.3}$$

Thus, if we know the orthogonal basis $\{\Phi_n^\circ\}_{n\in\mathbb{Z}}$, which is "close" to the set $\{\Phi_n\}_{n\in\mathbb{Z}}$, then we can check whether (1.3) is valid and conclude that we have (or have not) a Bari basis of the generalized eigenfunctions. However, sometimes it is quite difficult to "guess" the "closest" orthogonal basis. In such a case, we can use the inner characterization of a Bari basis (Gohberg and Krein 1996, Theorem 3.3). Namely, a complete set of vectors $\{\Phi_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ forms a Bari basis if and only if

- the set $\{\Phi_n\}_{n\in\mathbb{Z}}$ is *minimal*, i.e., no vector from the set belongs to a closed linear span of the remaining vectors of this set, and

- the matrix $\|(\Phi_n,\Phi_m)_{\mathcal{H}}-\delta_{nm}\|_1^\infty$ is squarable, i.e.,

$$\sum_{n,m}\|(\Phi_n,\Phi_m)_{\mathcal{H}}-\delta_{nm}\|^2<\infty. \tag{1.4}$$

Relation (1.4) allows us to decide whether we have a Bari basis without referring to any orthogonal basis. In addition, this relation gives us a hint how one can predict such important property of the generalized eigenfunctions as Bari basis property based on the results of numerical analysis. More precisely, we have in mind the following procedure. Consider a spectral problem for a Sturm–Liouville differential operator on a finite spatial interval. By using certain numerical method, we can reduce the problem of finding eigenvalues and eigenfunctions of a differential operator in a Hilbert space to the problem of finding $(2n+1)$ eigenvalues and eigenvectors of $(2n+1)\times(2n+1)$ matrix in the space $\mathbb{C}^{2n+1}$. Let $\{\lambda_m\}_{m\in\mathbb{Z}}$ and $\{\Phi_m\}_{m\in\mathbb{Z}}$ be the sets of the eigenvalues and eigenvectors of the operator, and $\{\mu_m\}_{|m|\le n}$ and $\{\Psi_m\}_{|m|\le n}$ be the eigenvalues and eigenvectors of the approximating matrix A_n. Let us select an increasing sequence of positive integers

$$n_1>n_2>n_3\ldots>n_p. \tag{1.5}$$

The approximating sequence of matrices A_{n_k}, $k=1,2,\ldots p$, has sequences of the eigenvalues $\{\mu_{m_k}\}_{m_k=1}^{n_k}$ and eigenvectors $\{\Psi_{m_k}\}_{m_k=1}^{n_k}$, $k=1,2,\ldots p$. It is very likely, that by increasing the size of the approximating matrix, we increase the accuracy of approximations for eigenvalues and eigenvectors of our main differential operator. Now let us construct the following finite sequence:

$$\{c_k\}_{k=1}^p:\ c_k=\sum_{m_k,l_k=1}^{n_k}\left\|\left(\Psi_{m_k},\Psi_{l_k}\right)_{\mathbb{C}^n}-\delta_{m_kl_k}\right\|^2. \tag{1.6}$$

If by examination, we observe that for large n_k, sequence $\{c_k\}$ changes very little (i.e., it is almost stable), then we have a good indication, that the set of the generalized eigenvectors of the main Sturm–Liouville operator forms a Bari basis. Definitely, the aforementioned analytical result requires rigorous mathematical proof. However, *the importance of predicting analytical facts using the results of numerical analysis is beyond any doubt*. The above described procedure will be carried out in our forthcoming paper.

Now we are in a position to outline briefly the content of the present paper.

In Section 2, we give a precise formulation of the initial boundary–value problem (see (2.1)–(2.4) below) and then rewrite this problem as the first order in time evolution equation (2.7). The dynamics generator of the evolution problem, which is a 2×2 matrix differential operator in a Hilbert space (the so–called energy space) is our main object of interest. In that Section, we reproduce the results of our works that are important for the current paper.

In Section 3, we introduce a nonselfadjoint quadratic operator pencil and indicate the connection between this pencil and the dynamics generator. By using the pencil, we derive an analytic function of the spectral parameter λ, the function whose roots coincide with the eigenvalues of the dynamics generator. The equation for the roots we call *the model equation*. In the same section, we introduce *the first approximation* and then *the refined second approximation* to the roots of the model equation.

In Section 4, we perform a numerical study of the roots of the model equation using Newton's method. We show the importance of accurate initial guesses in order to increase the accuracy of calculations. In particular, we have shown that using the refined approximation (the second one) to the characteristic equation yields extremely good result beginning from the very first eigenvalue (which is not the case for the first, more rough approximation).

Finally, in Section 5, we carry out the same program by using the Tchebychev spectral method.

In the conclusion of the Introduction, we would like to emphasize that numerical simulations provide excellent results when applied *to refined equation* (analytically a little more complicated). Our findings are summarized in the last Section – Conclusions.

2 Statement of problem. Auxiliary results from papers (Shubov $1999, 1998_1, 1997_{1,2}, 1996_{1,2}$).

Let us consider the following damped wave equation:

$$u_{tt} + 2\mathbf{d}(x)\, u_t + Lu = 0, \qquad x \in [0,a], \qquad a < \infty, \qquad t \geq 0, \tag{2.1}$$

where L is the Sturm–Liouville operator that acts on a smooth function φ by the formula

$$L\varphi = -\frac{1}{\rho(x)} \left(p(x)\, \varphi_x\right)_x + q(x)\, \varphi. \tag{2.2}$$

Eq. (2.1) describes the vibrations of a nonhomogeneous damped string with the following characteristics: $\mathbf{d}(\cdot)$ is the viscous damping coefficient, $\rho(\cdot)$ is the density of the string, $p(\cdot)$ is the modulus of elasticity and $q(\cdot)$ is the rigidity of an external harmonic force.

We consider the standard initial conditions

$$u(x,0) \quad = \quad u_0(x), \qquad u_t(x,0) = u_1(x) \tag{2.3}$$

and a two–parameter family of the boundary conditions

$$(u_x + ku_t)(0) = 0, \qquad (u_x + hu_t)(a) = 0, \qquad h,k \in \mathbb{C} \cup \{\infty\}. \tag{2.4}$$

For some particular values of (h,k), the boundary conditions (2.4) are well–known; e.g., if $k = h = \infty$, we have the homogeneous Dirichlet boundary conditions, and our problem describes the vibrations of a string with both ends fixed. If $k = h = 0$, we have the Neumann boundary conditions, and in that case we deal with a string whose both ends are free. If $k = -1$ and $h = 1$, we have the Sommerfeld radiation boundary conditions. In this case, the eigenvalues are called *resonances* and our problem describes the resonance phenomenon in acoustical scattering. In general, k and h are arbitrary complex numbers and it is a custom in current mathematical and engineering literature (see (Shubov $2004_{1,2,3}$) and references therein) to model the action of smart materials (self–straining actuators) by introducing parameters in the dynamical boundary conditions (2.4). We assume that the coefficients of Eq. (2.2) satisfy the following conditions:

$$\rho(x), p(x) \geq c > 0; \quad d(x), q(x) \geq 0; \quad \rho, p \in H^2[0,a], \quad \mathbf{d} \in H^1(0,a), \quad q \in L^\infty(0,a), \tag{2.5}$$

where $H^k(0,a)$, $k = 1,2$, are the standard Sobolev spaces (Adams 1995).

Remark 2.1. We mention here that the asymptotic behavior of the eigenvalues strongly depends on the behavior of the density function $\rho(\cdot)$ at the vicinity of the endpoint $x = a$. However, we can admit singularities and zeroes of ρ somewhere in the inner points of the interval $[0, a)$, even at the endpoint $x = 0$ (see (Shubov 1995) for the case $\mathbf{d} = 0$). The leading term of the asymptotic will be the same, but the remainder term will go to zero at the slower rate. We do not allow such a behavior for the sake of simplifi cation of the numerical analysis. (We are planning to address the question of singular density in our next paper.*)*

In order to introduce the nonselfadjoint operator, which will be our main object of interest, we have to make a standard reduction of the above initial boundary value problem to the fi rst order in time evolution equation in a Hilbert space, which will be called *the energy space*.

Let $\mathcal{H}$ be the space of a 2–component Cauchy data $U(x) = (u_0(x), u_1(x))^T$ equipped with the following energy norm (the superscript "T" means the transposition):

$$\|U\|^2_{\mathcal{H}} = \frac{1}{2}\int_0^a \left[p(x)\left|u_0'\right|^2 + q(x)\rho(x)|u_0|^2 + \rho(x)|u_1|^2\right] dx. \tag{2.6}$$

Problem (2.1)–(2.4) can be written as the following fi rst order in time evolution system in $\mathcal{H}$:

$$U_t = i\mathfrak{L}_{hk}U, \qquad U = \begin{pmatrix} u(x,t) \\ u_t(x,t) \end{pmatrix}, \qquad U|_{t=0} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \tag{2.7}$$

where the dynamics generator $\mathfrak{L}_{hk}$ is a matrix differential operator given by the following differential expression:

$$\mathfrak{L}_{hk} = -i\begin{pmatrix} 0 & 1 \\ -L & -2\mathbf{d}(x) \end{pmatrix} \tag{2.8}$$

defi ned on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}_{hk}) \;=\; & \Big\{\Phi = (\phi_0, \phi_1)^T \in \mathcal{H} : \phi_0 \in H^2(0,a), \quad \phi_1 \in H^1(0,a), \\ & \left(\phi_0' + k\phi_1\right)(0) = 0, \quad \left(\phi_0' + h\phi_1\right)(a) = 0\Big\}. \end{aligned} \tag{2.9}$$

From now on, the operator $\mathfrak{L}_{hk}$ is our main object of interest. We start with the statement summarizing the results proven in our aforementioned works.

Theorem 2.1. *(a) $\mathfrak{L}_{hk}$ is a simple, nonselfadjoint operator. (Recall that a nonselfadjoint operator is simple if the operator and its adjoint do not have an invariant subspace, on which they generate a selfadjoint operator.)*

(b) When $\mathrm{Re}\, h \geq 0$ *and* $\mathrm{Re}\, k \leq 0$*, then $\mathfrak{L}_{hk}$ is a dissipative operator, i.e.,* $\mathrm{Re}\,(\mathfrak{L}_{hk}\Phi, \Phi)_{\mathcal{H}} \geq 0$ *for all $\Phi \in \mathcal{D}(\mathfrak{L}_{hk})$.*

(c) When $\mathfrak{L}_{hk}$ is dissipative, it is a maximal operator, i.e., it does not admit other dissipative extensions.

(d) The adjoint operator $\mathfrak{L}_{hk}^*$ is given by the matrix differential expression

$$\mathfrak{L}_{hk}^* = -i \begin{pmatrix} 0 & 1 \\ -L & 2\mathbf{d}(x) \end{pmatrix} \tag{2.10}$$

defined on the domain

$$\begin{aligned} \mathcal{D}(\mathfrak{L}_{hk}^*) \;=\; & \Big\{ \Phi = (\phi_0, \phi_1)^T \in \mathcal{H} : \phi_0 \in H^2(0,a), \quad \phi_1 \in H^1(0,a), \\ & \left(\phi_0' - \overline{k}\phi_1\right)(0) = 0, \quad \left(\phi_0' - \overline{h}\phi_1\right)(a) = 0 \end{aligned} \tag{2.11}$$

(e) The operator $\mathfrak{L}_{hk}$ has a compact resolvent, which means that it has purely discrete spectrum consisting of the normal eigenvalues, i.e., each eigenvalue has a finite algebraic multiplicity and it is an isolated singularity of the resolvent.

One can see that in order to obtain the adjoint operator, it suffices to replace $\mathbf{d}$ with $(-\mathbf{d})$ and the parameters k and h with $\left(-\overline{k}\right)$ and $\left(-\overline{h}\right)$ respectively.

We note that the direct comparison of the formulas for $\mathfrak{L}_{hk}$ and $\mathfrak{L}_{hk}^*$ shows that there are obviously two reasons for the nonsubmissiveness:

(a) the difference in matrix differential expressions; (b) the distinction in the boundary conditions.

The second reason is relatively mild (for $\mathbf{d} = 0$, $k = \infty$, and $h = 1$ the spectral analysis is known). The combination of (a) and (b) makes the spectral analysis of the operator $\mathfrak{L}_{hk}$ rather nontrivial.

In order to formulate our main results for the operator $\mathfrak{L}_{hk}$, we have to recall some definitions. At the present moment, there is no general spectral theory of nonselfadjoint operators in a Hilbert space. There exists spectral theory only for some special classes of nonselfadjoint operators, one of which is the class of *spectral operators*. These operators were introduced by N. Dunford (Dunford and Schwartz 1963) (see, also, (Gohberg and Krein 1996)), and the whole 3rd volume of N. Dunford and J. Schwartz monograph (Dunford and Schwartz 1963) is devoted to this class of operators. The problem is that it is quite difficult to find nontrivial examples of such operators.

First, let us recall the definition of a *spectral operator*. We will not formulate the most general definition, but only the one which is sufficient for our purpose.

Definition 2.1. *Let $\{\psi_n\}_{n=1}^{\infty}$ be a Riesz basis in a complex Hilbert space H (a linear isomorphic image of an orthogonal basis) and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Define a linear operator S in H by the formula*

$$S\varphi = \sum_{n=1}^{\infty} \lambda_n (\varphi, \psi_n^*) \psi_n \tag{2.12}$$

on the domain

$$D(S) = \left\{ \varphi \in H : \sum_{n=1}^{\infty} |\lambda_n|^2 |(\varphi, \psi_n^*)|^2 < \infty \right\}. \tag{2.13}$$

S is called a scalar operator. Here by $\{\psi_n^*\}_{n=1}^{\infty}$ we denote the Riesz basis, which is biorthogonal to the basis $\{\psi_n\}_{n=1}^{\infty}$, i.e., $(\psi_n, \psi_m^*) = \delta_{nm}$ with δ_{nm} being the Kronecker symbol.

Definition 2.2. *An operator $\mathfrak{L}$ in H is called a spectral operator if it can be represented in the form*

$$\mathfrak{L} = S + N, \tag{2.14}$$

where S is a scalar operator, N is a bounded finite rank nilpotent operator (i.e., there exists an integer k such that $N^k = 0$), and N commutes with S.

In the general definition of a spectral operator, the spectral decomposition (2.14) may be continuous, which means that it may involve integration with respect to some spectral measure; N may be a generalized nilpotent operator, which means that it is bounded and its spectrum $\mathfrak{S}(N) = 0$.

Definition 2.2. can be reformulated as follows: $\mathfrak{L}$ in an operator whose matrix in an appropriate basis has a Jordan normal form with a finite number of nontrivial Jordan cells, and each cell has a finite size. To show that some specific operator $\mathfrak{L}$ is spectral in the sense of this definition, it is enough to show the following facts: (i) $\mathfrak{L}$ has purely discrete spectrum; (ii) $\mathfrak{L}$ may have a finite number of eigenvectors having finite chains of associated vectors; (ii) the system of root vectors of $\mathfrak{L}$ (eigenvectors and associated vectors together) forms a Riesz basis in H.

In practice, the most difficult part in the proof of the spectral property is (ii).

The most important result of our research (Shubov $1999, 1998_1, 1997_{1,2}, 1996_{1,2}$) is the following statement.

Theorem 2.2. *$\mathfrak{L}_{hk}$ is a spectral operator in the sense of Definition 2.2., i.e.,*

$$\mathfrak{L}_{hk} = S_{hk} + N_{hk}, \tag{2.15}$$

with the explicit formulas for S_{hk} and N_{hk}. (see Theorem 2.4. below).

Now we are in a position to formulate the statement on the spectral asymptotic.

Theorem 2.3. *1). $\mathfrak{L}_{hk}$ has purely discrete spectrum $\{\lambda_n\}_{n\in\mathbb{Z}}$. If the boundary parameters satisfy the conditions*

$$|h| \neq \sqrt{p(a)/\rho(a)}, \qquad |k| \neq \sqrt{p(0)/\rho(0)}, \tag{2.16}$$

then the entire spectrum is located in a strip parallel to the real axis

$$|\operatorname{Im}\lambda_n| \leq C < \infty. \tag{2.17}$$

If h, k are real, then $\lambda_{-n} = -\overline{\lambda}_n$.

2). The following asymptotic representation is valid for the eigenvalues:

$$\lambda_n = \Lambda_n^{hk} + O\left(|n|^{-1}\right), \quad |n| \to \infty,$$

$$\Lambda_n^{hk} = \mathcal{M}^{-1}\left[\left(n + \frac{\operatorname{sgn} n}{2}\right) + i\left(\mathcal{N} + \frac{1}{2}\ln\frac{\sqrt{p(a)/\rho(a)}+h}{\sqrt{p(a)/\rho(a)}-h} + \frac{1}{2}\ln\frac{\sqrt{p(0)/\rho(0)}-k}{\sqrt{p(0)/\rho(0)}+k}\right)\right], \tag{2.18}$$

where

$$\mathcal{M} = \int_0^a \sqrt{\rho(x)/p(x)}dx, \qquad \mathcal{N} = \int_0^a \mathbf{d}(x)\sqrt{\rho(x)/p(x)}dx. \tag{2.19}$$

Remark 2.2. If one of the conditions in (2.16) fails, then the spectrum will be totally different. Namely, the imaginary parts of the eigenvalues will not be bounded above; they will tend to infi nity at the rate $|\mathrm{Im}\,\lambda_n| \sim \ln|\mathrm{Re}\,\lambda_n|$ as $|n| \to \infty$. The consequence of such a behavior of the eigenvalues is quite serious, i.e., the eigenfunctions do not form an unconditional basis in that case. (For the derivation of the spectral asymptotic for the critical value of the boundary parameter, see our paper (Shubov 2002).)

Theorem 2.4. *1). All eigenvalues of the operator $\mathfrak{L}_{hk}$ have geometric multiplicities 1; i.e., for each λ_n, there exists only one eigenvector Ψ_n. However, a finite number of eigenvalues may have finite algebraic multiplicities m_n, i.e., for such λ_n, there exists a finite chain of associated vectors $\{\Psi_n^j\}_{j=1}^{m_n-1}$*

$$(\mathfrak{L}_{hk} - \lambda_n I)\,\Psi_n^j - \Psi_n^{j-1}, \qquad \Psi_n^0 \equiv \Psi_n, \qquad \Psi_n^{-1} = 0.$$

2). The entire set of the root vectors (eigenvectors and associated vectors together) of the operator $\mathfrak{L}_{hk}$ forms a Riesz basis in $\mathcal{H}$.

3). The corresponding scalar and nilpotent parts of $\mathfrak{L}_{hk}$ can be given explicitly. Let $\{\lambda_n,\ n \in R \subset \mathbb{Z}\}$ be the set of multiple eigenvalues with the multiplicities m_n, then

$$\begin{aligned} S_{hk}\varphi &= \sum_{n\in\mathbb{Z}\setminus R} \lambda_n\,(\varphi, \Psi_n^*)\,\Psi_n + \sum_{n\in R} \lambda_n \sum_{j=0}^{m_n-1} \left(\varphi, \Psi_n^{j*}\right)\Psi_n^j, \\ N_{hk}\varphi &= \sum_{n\in R} \sum_{j=1}^{m_n-1} \left(\varphi, \Psi_n^{j*}\right)\Psi_n^{j-1}, \quad \varphi \in D(\mathfrak{L}_{hk}), \end{aligned} \tag{2.20}$$

where Ψ_n^* and Ψ_m^{j*} are the vectors of the Riesz basis in $\mathcal{H}$, which is biorthogonal to the Riesz basis of the root vectors. (We recall that the biorthogonal Riesz basis is formed by the root vectors of the adjoint operator $\mathfrak{L}_{hk}^*$.)

3 Nonselfadjoint Operator Pencil Associated to Problem.

In this section, we introduce the nonselfadjoint polynomial operator pencil, which is induced by the main boundary value problem. There are different ways to introduce such a pencil, and we have chosen the one which is used in engineering literature. Let us look for a solution of the problem defi ned by Eq. (2.1) and conditions (2.3) and (2.4) in the form

$$\mathcal{V}(x,t) = e^{i\lambda t} u(x), \tag{3.1}$$

where λ is a complex parameter. Substituting (3.1) into Eq. (2.1) and conditions (2.3) and (2.4), we obtain that u has to satisfy the following boundary problem:

$$Lu + 2i\lambda \mathbf{d}(x)\,u - \lambda^2 u = 0, \tag{3.2}$$

$$(u_x + i\lambda k u)(0) = 0, \qquad (u_x + i\lambda h u)(a) = 0, \tag{3.3}$$

where L is given in (2.2).

Now we are in a position to define the following quadratic operator pencil. Let the pencil $\mathcal{P}_{hk}(\lambda)$ be defined by the formula

$$\mathcal{P}_{hk}(\lambda)\varphi = L\varphi + \lambda^2\varphi - 2i\lambda \mathbf{d}(x)\varphi, \quad (L \text{ is given in (2.2)}),$$

on the domain

$$D(\mathcal{P}_{hk}(\lambda)) = \left\{\varphi : \varphi \in H^2(0,a), \quad (\varphi' + i\lambda k\varphi)(0) = 0, \quad (\varphi' + i\lambda h\varphi)(a) = 0, \quad h,k \in \mathbb{C} \cup \{\infty\}\right\}. \tag{3.4}$$

Definition 3.1. *We say that $\lambda \in \mathbb{C}$ is an eigenvalue of the pencil $\mathcal{P}_{hk}(\lambda)$ if the problem $\mathcal{P}_{hk}(\lambda)\varphi = 0$ has a nontrivial solution. This solution is called an eigenmode or an eigenfunction.*

The pencil $\mathcal{P}_{hk}(\lambda)$ is closely related to the operator $\mathfrak{L}_{hk}$ given by (2.8), (2.9). Namely, both the operator $\mathfrak{L}_{hk}$ and pencil $\mathcal{P}_{hk}(\lambda)$ have the same spectra with the same multiplicities of the multiple eigenvalues (see (Shubov 2002)). Furthermore, let $\{F_n(x)\}_{n\in\mathbb{Z}}$ be the set of eigenfunctions of the pencil $\mathcal{P}_{hk}(\lambda)$, then the following relation between $\{F_n(x)\}_{n\in\mathbb{Z}}$ and $\{\Psi_n(x)\}_{n\in\mathbb{Z}}$ ($\Psi_n(x)$ is an eigenvector of $\mathfrak{L}_{hk}$) holds:

$$\Psi_n(x) = \begin{pmatrix} \frac{1}{i\lambda_n} F_n(x) \\ F_n(x) \end{pmatrix}, \qquad n \in \mathbb{Z}. \tag{3.5}$$

We mention here, that a finite number of the eigenvalues may have finite algebraic multiplicities that are greater that 1; for those eigenvalues we have finite chains of the associated vectors. We note, that formula (3.5) takes place only for the eigenfunctions of the operator $\mathfrak{L}_{hk}$ and the pencil $\mathcal{P}_{hk}(\lambda)$. The relation between the corresponding associated functions is more complicated and will not be used in this work.

Now we describe the adjoint pencil $\mathcal{P}^*_{hk}(\lambda)$ which is defined by the formula

$$\mathcal{P}^*_{hk}(\lambda)\varphi = L\varphi + \lambda^2\varphi + 2\mathbf{d}(x)\varphi, \tag{3.6}$$

on the domain

$$D(\mathcal{P}^*_{hk}(\lambda)) = \left\{\varphi : \varphi \in H^2(0,a), \quad (\varphi' - i\lambda\overline{k}\varphi)(0) = 0, \quad (\varphi' - i\lambda\overline{h}\varphi)(a) = 0\right\}. \tag{3.7}$$

The spectrum of the pencil $\mathcal{P}^*_{hk}(\lambda)$ coincides with the spectrum of the operator $\mathfrak{L}^*_{hk}$ given by (2.18), (2.19). Let $\{F^*_n(x)\}_{n\in\mathbb{Z}}$ be the eigenvectors of $\mathcal{P}^*_{hk}$, then the following relation between $\{F^*_n(x)\}_{n\in\mathbb{Z}}$ and the eigenvectors $\{\Psi^*_n(x)\}_{n\in\mathbb{Z}}$ of the operator $\mathfrak{L}^*_{hk}$ holds:

$$\Psi^*_n(x) = \begin{pmatrix} \frac{1}{i\lambda_n} F^*_n(x) \\ F^*_n(x) \end{pmatrix}, \qquad n \in \mathbb{Z}. \tag{3.8}$$

Remark 3.1. From now on, we will discuss the particular case of $k=\infty$, i.e., the left end boundary condition will be $u(0)=0$, which corresponds to the fixed left end of the string. All our results can be easily extended to any $k\in\mathbb{C}$. Our second simplification is more serious. Namely, in the present paper, we consider the string with constant parameters. The case of a string with variable density and/or viscous damping will be discussed in our forthcoming paper.

Let us consider the following boundary–value problem:

$$\rho^{-1}w''-2i\lambda\mathbf{d}w+\lambda^2 w=0\,, \tag{3.9}$$

$$w(0)=0,\qquad \left(w'+i\lambda hw\right)(a)=0. \tag{3.10}$$

The general solution of the differential equation (3.9) can be written in the form

$$w(\lambda,x)=A(\lambda)\,e^{ix\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}+B(\lambda)\,e^{-ix\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}, \tag{3.11}$$

with functions $A(\lambda)$ and $B(\lambda)$ to be determined from the boundary conditions.

The first boundary condition $u(\lambda,0)=0$ implies that $A(\lambda)=-B(\lambda)$. Inserting this result into the second boundary condition from (3.10) yields

$$\begin{aligned}\mathbf{0} &= u'(\lambda,a)+i\lambda hu(\lambda,a)\\ &= A(\lambda)\left[\sqrt{(\lambda^2-2i\lambda\mathbf{d})\,\rho}e^{ia\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}+\sqrt{(\lambda^2-2i\lambda\mathbf{d})\,\rho}e^{-ia\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}\right.\\ &\quad\left.+\lambda he^{ia\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}-\lambda he^{-ia\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}\right].\end{aligned} \tag{3.12}$$

From (3.12), it follows that λ is an eigenvalue of the problem (3.9), (3.10) if and only if this λ is a solution of the following transcendental equation:

$$e^{2ia\sqrt{(\lambda^2-2i\lambda\mathbf{d})\rho}}-\frac{\lambda h-\sqrt{(\lambda^2-2i\lambda\mathbf{d})\,\rho}}{\lambda h+\sqrt{(\lambda^2-2i\lambda\mathbf{d})\,\rho}}=0\,. \tag{3.13}$$

Using methods of asymptotic analysis, it can be shown that Eq. (3.13) has a countable set of roots, which cannot be found in a closed form. To find a convenient numerical scheme for the approximations of the aforementioned roots, we first introduce a simpler version of Eq. (3.13), which we call "the model equation." The roots of the model equation, which will be denoted as $\{\lambda_n^\infty\}_{n\in\mathbb{Z}}$, can be easily found. The importance of the model equation roots is that the points $\{\lambda_n^\infty\}_{n\in\mathbb{Z}}$ will be used as initial numerical values for an appropriate numerical scheme. To derive the model equation, let us introduce a new function

$$\varkappa(\lambda)=\sqrt{(\lambda^2-2i\mathbf{d}\lambda)\,\rho}. \tag{3.14}$$

Eq. (3.13) obtains the form

$$e^{2ia\varkappa(\lambda)}=\frac{\lambda h-\varkappa(\lambda)}{\lambda h+\varkappa(\lambda)}. \tag{3.15}$$

For $|\lambda| \to \infty$, we can approximate $\varkappa$ as follows:

$$\varkappa(\lambda) = \lambda\sqrt{\rho}\left(1 - \frac{2i\mathbf{d}}{\lambda}\right)^{1/2} = \lambda\sqrt{\rho}\left(1 - \frac{i\mathbf{d}}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right). \tag{3.16}$$

Therefore, with (3.16) the rational function in (3.15) can be approximated as

$$\begin{aligned}\frac{\lambda h - \varkappa(\lambda)}{\lambda h + \varkappa(\lambda)} &= \frac{\lambda\left(h - \sqrt{\rho}\right) - i\mathbf{d}\sqrt{\rho} + O\left(\lambda^{-1}\right)}{\lambda\left(h + \sqrt{\rho}\right) + i\mathbf{d}\sqrt{\rho} + O(\lambda^{-1})} \\ &= \frac{\lambda\left(h - \sqrt{\rho}\right) - i\mathbf{d}\sqrt{\rho}}{\lambda\left(h + \sqrt{\rho}\right) + i\mathbf{d}\sqrt{\rho}}\left(1 + O\left(\frac{1}{\lambda}\right)\right). \end{aligned} \tag{3.17}$$

Let us omit the remainder terms in (3.16) and (3.17) and substitute the leading terms into Eq. (3.15). We obtain a new equation

$$e^{2ia\sqrt{\rho}(\lambda - i\mathbf{d})} = \frac{\lambda\left(h - \sqrt{\rho}\right) - i\mathbf{d}\sqrt{\rho}}{\lambda\left(h + \sqrt{\rho}\right) + i\mathbf{d}\sqrt{\rho}}. \tag{3.18}$$

In what follows, Eq. (3.18) will be our *model equation.* It can be shown that the following formula is valid for the roots of Eq. (3.18):

$$\hat{\lambda}_n = \frac{\pi n}{a\sqrt{\rho}} + i\mathbf{d} + \frac{i}{2a\sqrt{\rho}}\ln\frac{h + \sqrt{\rho} + ia\mathbf{d}\rho\,(\pi n)^{-1}}{h - \sqrt{\rho} - ia\mathbf{d}\rho\,(\pi n)^{-1}} + O\left(\frac{1}{|n|}\right), \qquad |n| \to \infty. \tag{3.19}$$

Let us modify the logarithmic term in (3.19). We have

$$\begin{aligned}&\ln\frac{h + \sqrt{\rho} + ia\mathbf{d}\rho\,(a)\,(\pi n)^{-1}}{h - \sqrt{\rho} - ia\mathbf{d}\rho\,(a)\,(\pi n)^{-1}} \\ &= \ln\frac{h + \sqrt{\rho\,(a)}}{h - \sqrt{\rho\,(a)}} + \ln\left(1 + \frac{ia\mathbf{d}\rho}{\pi n\left(h + \sqrt{\rho}\right)}\right) - \ln\left(1 - \frac{ia\mathbf{d}\rho}{\pi n\left(h - \sqrt{\rho}\right)}\right) \\ &= \ln\frac{h + \sqrt{\rho}}{h - \sqrt{\rho}} + \frac{i\mathbf{d}a\rho}{\pi n\left(h + \sqrt{\rho}\right)} + \frac{ia\mathbf{d}\rho}{\pi n\left(h - \sqrt{\rho}\right)} + O\left(\frac{1}{n^2}\right) \\ &\approx \ln\frac{h + \sqrt{\rho}}{h - \sqrt{\rho}} + \frac{2ia\mathbf{d}\rho h}{\pi n\,(h^2 - \rho)}. \end{aligned} \tag{3.20}$$

The first and second approximations to the roots of the model equation. Let us take representation (3.19) and omit all terms of the order $O\left(|n|^{-1}\right)$. The remaining formula will be called the *first approximation* for the eigenvalues when the number of an eigenvalue tends to infi nity. So for the fi rst approximation, we have

$$\widetilde{\lambda}_n^{\infty} = \frac{\pi n}{a\sqrt{\rho}} + i\left[\mathbf{d} + \frac{1}{2a\sqrt{\rho}}\ln\frac{h + \sqrt{\rho}}{h - \sqrt{\rho}}\right]. \tag{3.21}$$

In addition to the first approximation given by (3.21), we also use the refined formula, which we will call *the second approximation*. That approximation can be obtained by incorporating formula (3.20) and it is given by the formula

$$\lambda_n^{\infty} = \left[\frac{\pi n}{a\sqrt{\rho}} - \frac{2a\mathbf{d}\rho h}{\pi n\left(h^2-\rho\right)}\right] + i\left[\mathbf{d} + \frac{1}{2a\sqrt{\rho}}\ln\frac{h+\sqrt{\rho}}{h-\sqrt{\rho}}\right]. \tag{3.22}$$

Remark 3.2. In the present paper, we will discuss the approximations for eigenvalues associated to both formulas (3.21) and (3.22). We show that even though the second approximation looks more complicated, it gives a much better numerical result.

4 Numerical Approximation for Eigenvalues Using Characteristic Equation.

In this section, we use the nonlinear characteristic equation (3.13) in order to approximate the eigenvalues numerically. Our goal is to examine how fast the eigenvalues converge to the asymptotic distributions given by formulas (3.21) and (3.22). These two formulas were obtained by omitting the remainder terms of order $O\left(|n|^{-1}\right)$. We will determine the actual numerical ranges for those terms and, most importantly, we look at the size of the error for small values of n.

The numerical results are obtained by using Newton's method, which is still considered as the most widely used, and one of the most efficient *root finding algorithm*s.

Let us introduce the following function of a complex variable z:

$$F(z) = e^{2ia\sqrt{\left(z^2-2iz\mathbf{d}\right)\rho}} - \frac{zh-\sqrt{\left(z^2-2iz\mathbf{d}\right)\rho}}{zh+\sqrt{\left(z^2-2iz\mathbf{d}\right)\rho}}. \tag{4.1}$$

It is known that the iterative formula

$$z_{i+1} = z_i + \frac{F(z_i)}{F'(z_i)}, \tag{4.2}$$

can be used for complex–valued functions as well. In our particular setting, we chose not to look for more sophisticated root finding methods, for example, the damped Newton's method or Muller's method with deflation (Muller 1956). On one hand, we have obtained good results with the simple Newton's method when we started with the refined approximations of the second type (3.22). On the other hand, we introduce another numerical method in Section 5, the Tchebychev method with the QT algorithm for the approximation of the eigenvalues. This later method does not use the model equation and does not require initial guesses.

All numerical simulations have been performed for parameter values $a = 2$, $\rho = 4$, $d = 2$, and $h = 4$. The explanation below indicates no qualitative change for different parameters values.

4.1 Using the asymptotic distribution.

As most numerical methods for finding roots of functions, Newton's method requires starting points for the search. A natural choice is to use explicit formulas (3.21) or (3.22) in order to

generate initial guesses. While the second approximations (3.22) of the asymptotic are expected to be more precise for eigenvalues located far from the origin, it is not necessarily true for eigenvalues that are close to the origin. Since we are especially interested in these eigenvalues, we use both approximations.

Initial guesses of the first type. We use Newton's method with first approximations (3.21) of the asymptotic as initial guesses. The results of calculations are presented in Tables 1 and 2 below.

Table 1.

1. In Table 1, we have listed the approximations of the eigenvalues using the first forty asymptotic eigenvalues $\left\{\widetilde{\lambda}_n^{\infty}\right\}_{n=1}^{40}$ as the initial guesses. It can be seen that the first ten approximations behave erratically. Namely,

   (a) We have found the same eigenvalues several times starting from different initial guesses (see rows # 1, 2, 3, and 8 of Table 1).

   (b) In two cases (rows # 7 and 9), Newton's method did not reach convergence at all. In fact, numerical overflow occurred in evaluating the characteristic function and its derivative.

   (c) Eigenvalues were not found in the order of increasing real parts, or in any other logical order, although the initial guesses were listed in the order of increasing real parts.

Table 2.

2. In Table 2, we have listed the first 25 eigenvalues ordered by increasing real parts. Although formula (3.21) provides us with even distance between the real parts of consecutive eigenvalues, the distribution of the first ten eigenvalues is quite irregular. At this point, it is not clear whether this behavior is the result of the asymptotic formula (3.21) being invalid for eigenvalues close to the origin, or the result of the irregular convergence properties of Newton's method.

Figure 1.

3. After the first ten steps, the remaining approximations lay close to the asymptotic expressions. As Figure 1 shows, the absolute value of the difference between the asymptotic eigenvalues $\left\{\widetilde{\lambda}_n^{\infty}\right\}$ and the true eigenvalues $\{\lambda_n\}$ (found numerically with high precision) decreases as n increases.

Explanation of the irregularities in Tables 1 and 2. The irregular results of Newton's method can be explained by taking into account the form of the characteristic equation (3.13). The main feature of the characteristic equation is that it contains an exponential term. Both the real and the imaginary parts of the function $F(z)$ are oscillating in the direction of the real axis and they both decrease exponentially along the imaginary axis (see Figure 2). These properties seriously

affect numerical calculations. The oscillations as well as the exponentially small values introduce zero and "near zero" components in the gradient of the characteristic function. As a result, the numerical iteration can jump suddenly from one point to another point far away. Even in the case of convergence, the solution (if found) can be very far from the initial guess. Starting with small initial guess, we have obtained large eigenvalues in some cases. Starting with large initial guess leads to finding small eigenvalues, occasionally filling in some gaps between previously found eigenvalues. The same eigenvalue was found several times (suggesting the existence of eigenvalues with multiplicity greater than one) even when the starting points were far away from each other. In each of those cases the derivative of the characteristic function showed that there are no multiple eigenvalues.

Figure 2.

Initial guesses of the second type. We now use approximations (3.22) of the asymptotic as our initial guesses. We leave out the first two terms of this sequence as they are negative for the given values of parameters. In Table 3, we have listed the approximations for the eigenvalues using the first forty asymptotic eigenvalues $\{\lambda_n^\infty\}$ as the initial guesses. When we used the refined initial guesses of Eq. (3.22), convergence from all of the initial guesses to the closest eigenvalue follows in at most seven steps with a precision of at least 10^{-10}. As Figure 3 shows, the absolute value of the difference between the asymptotic eigenvalues and the true eigenvalues is decreasing for increasing n.

Figure 3.

Comparison of the results based on initial guesses. In Figure 4, we compare the convergence rate of eigenvalues to the approximate asymptotic values. We know from (3.19), (3.21), and (3.22) that the distance between the approximate asymptotic values and the actual eigenvalues is $O\left(|n|^{-1}\right)$. In order to quantify this remainder term more precisely, we have multiplied it by n and plotted the result in Figure 4. According to this figure, the remainder term in (3.21) can be approximated as $O\left(|n|^{-1}\right) \approx 2.32\,|n|^{-1}$ and in (3.22) as $O\left(|n|^{-1}\right) \approx 1.34\,|n|^{-1}$.

Figure 4.

4.2 Approximation of eigenvalues without knowledge of good initial guesses.

As is well–known, the use of asymptotic eigenvalues as initial guesses is questionable when one looks for eigenvalues close to the origin. It is important to investigate whether there are eigenvalues close to the origin other than the ones listed in Tables 1 and 3. For this purpose, we create a grid of points in a rectangular region of the complex plane and take all the points as initial conditions for Newton's method. As it is well–known, Newton's method converges very fast if it converges at all. In our case, due to the exponential term in the characteristic function, it also diverges very fast when it diverges. Hence the method of checking relatively large number of points of the complex plain whether they are zeros of the characteristic function is quite feasible numerically.

Figure 5 shows the convergence properties of Newton's method applied to the characteristic equation. Namely, the points of the plain are colored differently depending on the different number of iterations needed for convergence when starting the iteration from the given points. Dark red areas correspond to failure of convergence. The darkest blue points correspond to fixed points of the iteration, i.e. they are precisely the eigenvalues.

Figure 5.

Figure 5 also shows that the distribution of eigenvalues is quite regular from the very beginning as it was observed in Table 3 already. Another important discovery revealed in this figure is the presence of five eigenvalues positioned on the imaginary axes. Figure 6 shows a closer look at this part of the complex plane. Choosing better initial guesses from the fractal images, Newton's method can now find more precise approximations of the eigenvalues. Table 4 lists the pure imaginary eigenvalues, Table 5 lists the first ten complex eigenvalues and their distance from each other.

Figure 6.

Table 4.

Table 5.

5 Numerical Approximations of Eigenvalues by Chebyshev's Method.

In this section, we use Chebyshev's method (Dongara et. al. 1996) to obtain a finite dimensional approximation of the spectral problem

$$-i\begin{pmatrix} 0 & 1 \\ \frac{1}{\rho}\mathrm{d}^2/\mathrm{d}x^2 & -2\mathbf{d} \end{pmatrix}\begin{pmatrix} w \\ w_1 \end{pmatrix} = \lambda\begin{pmatrix} w \\ w_1 \end{pmatrix}. \tag{5.1}$$

$$w(0) = 0, \qquad \left(w' + i\lambda h w\right)(a) = 0. \tag{5.2}$$

This method became increasingly popular during the last two decades in computational fluid dynamics (see, e.g., (Dongara et. al. 1996) and references therein) for its simplicity in discretizing boundary value problems and its precision in determining a large number of eigenvalues.

Remark 5.1. While it is possible to measure the discretization error by using the so called Tchebychev–Tau method, we will not do the "Tau" part here. Instead, we measure to total error (the error from the discretization and the error from the numerical solution of the finite dimensional eigenvalue problem) by evaluating the known characteristic function (4.1) at the numerical approximations of the eigenvalues.

Using the notation $\psi = (\psi_0, \psi_1)^T = (w, w_1)^T$, where the superscript "T" means the transposition, we can formulate eigenvalue problem (5.1) as

$$\mathcal{L}_h \psi = \lambda \psi, \tag{5.3}$$

where the differential operator $\mathcal{L}_h$ is given by the formula

$$\mathcal{L}_h = -i \begin{bmatrix} 0 & 1 \\ \frac{1}{\rho} \mathrm{d}^2/\mathrm{d}x^2 & -2\mathbf{d} \end{bmatrix} \tag{5.4}$$

defi ned on the domain

$$\mathcal{D}(\mathcal{L}_h) = \left\{ \psi \in \mathcal{H} : \psi_0 \in H^2(0,a), \psi_1 \in H^1(0,a), \psi_0(0) = 0, \psi_0'(a) + h\psi_1(a) = 0 \right\}. \tag{5.5}$$

In our numerical approach, we will use Chebyshev's polynomials [17], which requires the transformation from the interval $[0,a]$ to the interval $[-1,1]$. Let us use the standard coordinate transformation

$$\psi_{\text{new}}(x) = \psi_{\text{old}}\left(\frac{1}{2}ax + \frac{1}{2}a \right), \quad x \in [-1,1], \tag{5.6}$$

whose inverse is

$$\psi_{\text{old}}(x) = \psi_{\text{new}}\left(\frac{x - \frac{1}{2}a}{\frac{1}{2}a} \right), \quad x \in [0,a]. \tag{5.7}$$

The transformed operator has the form

$$\widetilde{\mathcal{L}}_h = -i \begin{bmatrix} 0 & 1 \\ \frac{1}{\widetilde{\rho}} \mathrm{d}^2/\mathrm{d}x^2 & -2\mathbf{d} \end{bmatrix}, \qquad \widetilde{\rho} = \left(\frac{a}{2} \right)^2 \rho, \tag{5.8}$$

and the transformed domain has the form

$$\begin{aligned} \mathcal{D}\left(\widetilde{\mathcal{L}}_h \right) \;=\; & \Big\{ \psi \in \widetilde{\mathcal{H}} : \psi_0 \in H^2(-1,1), \psi_1 \in H^1(-1,1), \\ & \psi_0(-1) = 0, \psi_0'(1) + \widetilde{h}\psi_1(1) = 0 \Big\}, \qquad \widetilde{h} = \frac{a}{2}h. \end{aligned} \tag{5.9}$$

The new state space $\widetilde{\mathcal{H}}$ is a Hilbert space, whose metric coincides with the metric of Hilbert space $\mathcal{H}$ up to replacement of ρ with $\widetilde{\rho}$ in formula (2.6).

5.1 Reduction of the second order derivative.

In numerical calculations, the approximation of higher order derivatives results in larger errors than that of the lower order derivatives, and therefore, it is desirable to reduce the second order system into a somewhat larger system that contains the fi rst order derivatives only. The latter reduction can be done by introducing new functions $\psi_2 = \psi_0'$ and $\Psi = (\psi_0, \psi_1, \psi_2)^T$.

We obtain the following generalized eigenvalue problem which is now equivalent to (5.3):

$$L_h \Psi = \lambda B \Psi, \tag{5.10}$$

where

$$L_h = -i \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2\mathbf{d} & \frac{1}{\widetilde{\rho}} \frac{d}{dx} \\ \frac{d}{dx} & 0 & -1 \end{bmatrix}, \tag{5.11}$$

$$\begin{aligned} \mathcal{D}(L_h) &= \Big\{ \Psi \in \widetilde{\mathcal{H}} : \psi_0 \in H^2(-1,1), \psi_1, \psi_2 \in H^1(-1,1), \\ &\quad \psi_0(-1) = 0, \psi_2(1) + \widetilde{h}\psi_1(1) = 0 \Big\}, \end{aligned} \tag{5.12}$$

and B is a singular matrix given by the formula $B = \mathrm{diag}\{1,1,0\}$.

5.2 Discretization by Chebyshev's method.

We approximate $\Psi = (\psi_0, \psi_1, \psi_2)^T \in \widetilde{\mathcal{H}}$ and its derivatives with Chebyshev's polynomials of orders up to either $(n+1)$ or $(n+2)$, depending on the boundary conditions involving different components of Ψ. Let ψ_0, ψ_1, and ψ_2 be approximated by

$$\psi_0(x) \approx \sum_{j=0}^{n+1} v_{0j} T_j(x), \quad \psi_1(x) \approx \sum_{j=0}^{n} v_{1j} T_j(x), \quad \psi_2(x) \approx \sum_{j=0}^{n+1} v_{2j} T_j(x), \qquad x \in [-1,1], \tag{5.13}$$

where $T_j(x)$ denotes the j^{th} Tchebychev polynomial defined by

$$T_j(x) = \cos(j \arccos x), \qquad j = 0,1,\ldots, \quad x \in [0,1].$$

The finite dimensional approximation (5.13) can be written now in a vector form using only Chebyshev's coefficients $\{v_{ij}\}$ from (5.13)

$$\Psi = (\psi_0, \psi_1, \psi_2)^T \longleftrightarrow \vec{v} = (v_{00}, v_{01}, \ldots v_{0,n+1}, v_{10}, v_{11}, \ldots v_{1n}, v_{20}, v_{21}, \ldots, v_{2,n+1})^T. \tag{5.14}$$

For the approximation of the first derivative of a function $\Psi \in \widetilde{\mathcal{H}}$, we will use the notation

$$\Psi' \approx \left(v'_{00}, v'_{01}, \ldots v'_{0,n+1}, v'_{10}, v'_{11}, \ldots v'_{1n}, v'_{20}, v'_{21}, \ldots, v'_{2,n+1}\right)^T. \tag{5.15}$$

The *derivative* matrix that will approximate differentiation in (5.11) can be calculated in several ways. We use the recurrence relation (Cavuto et.al. 1988)

$$c_j v'_{ij} = v'_{i,j+2} + 2(j+1) v_{i,j+1} \qquad 0 \le j \le n+1, \quad i = 0,1,2, \tag{5.16}$$

where

$$c_j = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j \ge 1, \end{cases} \tag{5.17}$$

and

$$v'_{ij} = 0, \qquad j \ge n, \quad i = 0,1,2. \tag{5.18}$$

From Eq. (5.16), we obtain the formula

$$v'_{ij} = \frac{2}{c_j} \sum_{\substack{k=j+1 \\ k+j \text{ is odd}}}^{n} k v_{ik}, \quad j = 0,1,\ldots,n+1, \quad i = 0,1,2. \tag{5.19}$$

Formula (5.19) results in the following discrete differentiation matrix of size $(n+1)\times(n+2)$:

$$D_{(n+1)\times(n+2)} = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 5 & & \\ 0 & 0 & 4 & 0 & 8 & 0 & & \\ 0 & 0 & 0 & 6 & 0 & 10 & & \\ & & & \ddots & \ddots & & & \\ & & & & & & & \\ 0 & & & \cdots & & & 0 & 2(n+1) \end{pmatrix}_{(n+1)\times(n+2)}. \tag{5.20}$$

From now on, we will use the above introduced subscript notation in order to specify the size of matrices. We will omit this subscript notation in order to get more compact formulas once the sizes of matrices are explained in detail. Note that the matrix $D_{(n+1)\times(n+2)}$ from (5.20) has the number of rows one unit less than the number of columns because we have omitted the last row, which would have zero entries everywhere due to the truncation in the Chebyshev's approximations (5.13). The first column of the matrix $D_{(n+1)\times(n+2)}$ also contains only zeros, but this part cannot be omitted as it represents the derivative of the constant terms in the Chebyshev's expansions (5.13). Once we obtain two equations for the boundary values in terms of Chebyshev's polynomials, the discredited eigenvalue problem will have the form

$$A\vec{v} = \lambda B\vec{v}, \tag{5.21}$$

where

$$A = -i \begin{pmatrix} 0_{(n+1)\times(n+2)} & I_{(n+1)\times(n+1)} & 0_{(n+1)\times(n+2)} \\ 0_{(n+1)\times(n+2)} & -2\mathbf{d}I_{(n+1)\times(n+1)} & \frac{1}{\widetilde{\rho}}D_{(n+1)\times(n+2)} \\ D_{(n+1)\times(n+2)} & 0_{(n+1)\times(n+1)} & -I_{(n+1)\times(n+2)} \\ \{\text{two} & \text{boundary} & \text{conditions}\} \end{pmatrix} \tag{5.22}$$

and

$$B = \begin{pmatrix} I_{(n+1)\times(n+2)} & 0_{(n+1)\times(n+1)} & 0_{(n+1)\times(n+2)} \\ 0_{(n+1)\times(n+2)} & I_{(n+1)\times(n+1)} & 0_{(n+1)\times(n+2)} \\ 0_{(n+1)\times(n+2)} & 0_{(n+1)\times(n+1)} & 0_{(n+1)\times(n+2)} \\ 0_{2\times(n+2)} & 0_{2\times(n+1)} & 0_{2\times(n+2)} \end{pmatrix}. \tag{5.23}$$

Matrices A and B are of the same sizes $(3n+5)\times(3n+5)$ and they both contain several zero submarines $\mathbf{0}$ and identity submarines $\mathbf{I}$ of appropriate sizes. Obviously, B is singular with $(n+3)$ rows containing zeros only.

The boundary conditions $\psi_0(-1) = 0$ and $\psi_2(1) + \widetilde{h}\psi_1(1) = 0$ from (5.12) require the evaluation of Chebyshev's polynomials at the end points $x = -1$ and $x = 1$. It is known that $T_j(\pm 1) = (\pm 1)^j$, where T_j is the j^{th} Tchebychev polynomial for $j \in \mathbb{N}$. With these boundary values, we

obtain the following two equations for the vector of Tchebychev coeffi cients $\vec{v}$:

$$0 \quad = \quad \psi_0(-1) = \sum_{j=0}^{n+1} v_{0j}(-1)^j \tag{5.24}$$

and

$$0 \quad = \quad \psi_2(1) + \widetilde{h}\psi_1(1) = \sum_{j=0}^{n+1} v_{2j}T_j(1) + \widetilde{h}\sum_{j=0}^{n} v_{1j}T_j(1) = \sum_{j=0}^{n+1} v_{2j} + \widetilde{h}\sum_{j=0}^{n} v_{1j}. \tag{5.25}$$

Eqs. (5.24) and (5.25) constitute the last 2 rows of the matrix A from (5.22).

Rigorously speaking, the Tchebyshev's spectral method means obtaining a fi nite dimensional eigenvalue problem of the form (5.21) through the following steps:

1. Take the fi nite dimensional approximations (5.13) of the functions $\psi_0 \in H^2(-1,1)$, $\psi_1, \psi_2 \in H^1(-1,1)$ and substitute the truncated Chebyshev's series (5.13) into the infi nite dimensional eigenvalue problem (5.10).

2. Take the weighted $L_2(-1,1)$–inner product (with weight function $1/\sqrt{1-x^2}$) of equation (5.10) and Chebyshev's polynomials T_j, for $j = 0,1,\ldots,n$.

3. Incorporate the boundary conditions into the matrices in the above described way.

Due to the orthogonality of the Tchebyshev's polynomials in the aforementioned weighted $L_2(-1,1)$ space, the resulting fi nite dimensional eigenvalue problem will have the same row entries as eigenvalue problem (5.21) up to constant multipliers either π or $\pi/2$.

5.3 Solving the Generalized Matrix Eigenvalue Problem (5.21).

We recall that our goal is to solve equation (5.21) with matrices A and B given by (5.22) and (5.23) respectively. Matrix A is a complex nonsymmetric matrix, while matrix B is a real nonsymmetric, nonsingular matrix. We use the IMPS FORTRAN subroutine DIVING (IMSL 1997, p.841)imps to compute the eigenvalues of (5.21). This subroutine is based on the iterative QT algorithm (Moller and Stewart 1973), which avoids the inversion of matrix B. We recall that the main output of the subroutine is a pair of complex vectors (α, β). In case of a nonzero $\beta(i)$ entry a generalized eigenvalue is obtained by the quotient $\alpha(i)/\beta(i)$.

This method clearly requires so–called *postprocessing*, where we have to analyze the numerical results for correctness and convergence. First of all, we have to exclude the $(\alpha(i), \beta(i))$ pairs with zero $\beta(i)$ values as they are the reflections of the zero rows of matrix B. Then, in the case when both $\alpha(i)$ and $\beta(i)$ are very small but nonzero numbers, the validity of taking their quotient is always questionable numerically. This step might lead to the so–called spurious eigenvalues that are not necessarily large. We exclude these eigenvalues by checking convergence properties as the discretization is refi ned. Table 6 shows very good convergence properties for all eigenvalue candidates except for number six. The results in place number six are spurious eigenvalues. The

first one hundred convergent values agreed with each other up to seven decimal places for discretization $n = 200$ and $n = 300$. The last column of Table 6 shows that these numbers satisfy the characteristic equation with precision up to 10^{-10}.

Table 6.

6 Conclusion.

In this paper we have investigated the distance between eigenvalues and their asymptotic estimates for a boundary–value problem for a damped string model. The exact eigenvalues were approximated by two very different numerical methods.

First we approximated the solutions to the characteristic equation using Newton's method. Two formulas for the asymptotic eigenvalues were used as initial guesses. We found that initial guesses obtained by the formula with increased precision in the real values were all in the basin of attraction and they resulted in quite good approximations of the eigenvalues even around the origin, except for purely imaginary eigenvalues.

Next we have chosen the Tchebychev's discretization of the problem combined with the QT algorithm. This resulted in solving a generalized matrix eigenvalue problem. Although the right hand side matrix is singular, the QT algorithm proved to be quite effective, the one spurious eigenvalue was easy to detect, and we obtained good approximation of a large number of eigenvalues.

In our forthcoming papers, we will discuss the dependence of the eigenvalues on the parameters and examine various damped wave equation with spatially nonhomogeneous coefficients. We will also discuss *the problem of numerical construction of control laws* for distributed and/or boundary control problem related to the damped wave equation.

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References

[1] Shubov, Marianna A., Asymptotics of resonances and eigenvalues for nonhomogeneous damped string, *Asympt. Anal.*, **13**, (1996), p.31–78.

[2] Shubov, Marianna A., Asymptotics of spectrum and eigenfunctions for nonselfadjoint operators generated by radial nonhomogeneous damped wave equations, *Asympt. Anal.*, **16**, (1998), p.245–272.

[3] Shubov, Marianna A., Nonselfadjoint operators generated by the equation of nonhomogeneous damped string, *Trans. Amer. Math. Soc.*, **349**, (1997), p.4481–4499.

[4] Shubov, Marianna A., Spectral operators generated by damped wave equations, *Integ. Eqs. Oper. Theory,* **28**, (1997), p.358–372.

[5] Shubov, Marianna A., Basis property of eigenfunctions of nonselfadjoint operator pencils generated by equation of nonhomogeneous damped string, *Integ. Eqs. Oper. Theory,* **25**, (1996), p.289–328.

[6] Shubov, Marianna A., The Riesz basis property of the system of the root vectors of a nonhomogeneous damped string. Transformation operators method, *Meth. Analysis Appl.*, **6**, (4), (1999), p.571–592.

[7] Shubov, Marianna A., Exact boundary and distributed controllability of radial damped wave equation, *J. Mathematiques Pures et Appl.*, **77**, (1998), p.415–437.

[8] Shubov, Marianna A., Martin, C.F., Dauer, J.P., Belinskiy, B.P., Unique controllability of damped wave equation, *SIAM J. Control Optim.,* **35**, (5), (1997), p.1773–1789.

[9] Shubov, Marianna A., Martin, C.F., Boundary and distributed controllability of the damped wave equation. Reduction of control time, *J. Math. Analysis Appl.,* **240**, (1999), p.16–36.

[10] Gohberg I. Ts. and Krein M.G., *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Math. Monogr., **18**, AMS, Providence, RI, (1996).

[11] Shubov, Marianna A., Flutter phenomenon and its mathematical analysis: Part I, II, and III; an invited survey paper for *Journal of Aerospace Engineering*.

[12] Adams R.A., *Sovolev Spaces*, Academic Press, New York, (1975).

[13] Shubov, Marianna A., Asymptotics of resonances and geometry of resonance states in the scattering of acoustic waves by inhomogeneity of density, *Differ. Integral Eqs.,* **8**, (3) (1995), p.1073–1115.

[14] Dunford N., Schwartz J.T., *Linear Operators, Part III, Spectral Operators*, Interscience Publishers, (1963).

[15] Shubov, Marianna A., Asymptotic and spectral analysis of nonselfadjoint operators generated by a fi lament with a critical value of a boundary parameter. *Math. Methods Appl. Sciences,* **26**, (2002), p.213–245.

[16] Canuto, C., Hussaini, M.Y., Quarteroni, A., and Zang, T.A., *Spectral Methods in Fluid Dynamics*, Springer Series in Computational Physics, Springer-Verlag, New York, (1988).

[17] Dongarra, J.J., Straughan, B., and Walker, D.W., Chebyshev tau–QZ algorithm methods for calculating spectra of hydrodynamic stability problems, *Appl. Numer. Math.*, **22**, (4), (1996), p.399–434.

[18] IMSL Fortran Subroutines for Mathematical Applications, Vol. 1&2, Visual Numerics, Inc., 1997.

[19] C.B. Moler and G.W. Stewart, An algorithm for generalized matrix eigenvalue problems, *SIAM Journal on Numerical Analysis*, 10/2, (1973), 241–256.

[20] D.E. Müller, A method for solving algebraic equations using an automatic computer, *Mathematical Tables and Aids to Computation*, 10, (1956), 208–215.

#	Initial Guess	Root	$\|F(z)\|$	$\|F'(z)\|$	Iter.
1	0.0000000 + 2.1373265i	5.1805941 + 2.1281372i	4.5422164E-13	2.6723035	19
2	0.7853982 + 2.1373265i	5.1805941 + 2.1281372i	4.4464397E-15	2.8926347	32
3	1.5707963 + 2.1373265i	1.4278082 + 2.1057120i	7.0120055E-15	2.9124502	5
4	2.3561945 + 2.1373265i	2.5317830 + 2.1153793i	1.7554167E-16	3.0975499	6
5	3.1415927 + 2.1373265i	5.1805941 + 2.1281372i	1.5313433E-15	2.8926347	20
6	3.9269908 + 2.1373265i	17.1818465 + 2.1362522i	5.9369305E-13	2.6941650	41
7	4.7123890 + 2.1373265i	55.6867227 - 164.97757i	NaN	NaN	24
8	5.4977871 + 2.1373265i	1.4278082 + 2.1057120i	3.9252311E-16	2.9124502	20
9	6.2831853 + 2.1373265i	133.1377161 + 46.6157834i	NaN	NaN	6
10	7.0685835 + 2.1373265i	40.0136606 + 2.1371236i	8.4034608E-13	2.6718718	157
11	7.8539816 + 2.1373265i	7.6369533 + 2.1324590i	6.6813782E-14	2.7895566	8
12	8.6393798 + 2.1373265i	8.4428388 + 2.1332475i	2.7950486E-13	2.7700081	7
13	9.4247780 + 2.1373265i	9.2451439 + 2.1338619i	2.4184011E-15	2.7546587	7
14	10.2101761 + 2.1373265i	10.0447401 + 2.1343492i	5.8291372E-12	2.7424194	6
15	10.9955743 + 2.1373265i	10.8422355 + 2.1347418i	4.2525102E-14	2.7325215	6
16	11.7809725 + 2.1373265i	11.6380675 + 2.1350624i	8.4686919E-15	2.7244146	6
17	12.5663706 + 2.1373265i	12.4325594 + 2.1353274i	5.8054288E-15	2.7176978	6
18	13.3517688 + 2.1373265i	13.2259554 + 2.1355489i	2.8296274E-10	2.7120746	5
19	14.1371669 + 2.1373265i	14.0184435 + 2.1357358i	4.3085185E-11	2.7073226	5
20	14.9225651 + 2.1373265i	14.8101707 + 2.1358950i	7.2837630E-12	2.7032725	5
21	15.7079633 + 2.1373265i	15.6012539 + 2.1360315i	1.3510469E-12	2.6997938	5
22	16.4933614 + 2.1373265i	16.3917870 + 2.1361495i	2.6768488E-13	2.6967848	5
23	17.2787596 + 2.1373265i	17.1818465 + 2.1362522i	6.9014762E-14	2.6941650	5
24	18.0641578 + 2.1373265i	17.9714953 + 2.1363421i	1.8169374E-14	2.6918707	5
25	18.8495559 + 2.1373265i	18.7607855 + 2.1364212i	3.3003542E-15	2.6898503	5
26	19.6349541 + 2.1373265i	19.5497608 + 2.1364912i	3.8548288E-15	2.6880622	5
27	20.4203522 + 2.1373265i	20.3384582 + 2.1365534i	4.3058059E-15	2.6864723	5
28	21.2057504 + 2.1373265i	21.1269087 + 2.1366089i	3.7975470E-15	2.6850525	5
29	21.9911486 + 2.1373265i	21.9151393 + 2.1366587i	6.2955044E-15	2.6837793	5
30	22.7765467 + 2.1373265i	22.7031730 + 2.1367035i	9.1615685E-15	2.6826335	5
31	23.5619449 + 2.1373265i	23.4910295 + 2.1367439i	4.8130171E-15	2.6815986	5
32	24.3473431 + 2.1373265i	24.2787263 + 2.1367805i	7.5122644E-10	2.6806608	4
33	25.1327412 + 2.1373265i	25.0662785 + 2.1368138i	4.4890257E-10	2.6798084	4
34	25.9181394 + 2.1373265i	25.8536992 + 2.1368442i	2.7261639E-10	2.6790313	4
35	26.7035376 + 2.1373265i	26.6410003 + 2.1368719i	1.6809829E-10	2.6783210	4
36	27.4889357 + 2.1373265i	27.4281920 + 2.1368974i	1.0514450E-10	2.6776700	4
37	28.2743339 + 2.1373265i	28.2152835 + 2.1369207i	6.6662752E-11	2.6770719	4
38	29.0597320 + 2.1373265i	29.0022830 + 2.1369422i	4.2802947E-11	2.6765212	4
39	29.8451302 + 2.1373265i	29.7891978 + 2.1369620i	2.7823002E-11	2.6760130	4
40	30.6305284 + 2.1373265i	30.5760345 + 2.1369804i	1.8286507E-11	2.6755430	4

Table 1: First forty approximations with initial guesses of the first type (3.21).

#	Root	Difference		$\lvert F(z)\rvert$	$\lvert F'(z)\rvert$	Iter.
		Real	Complex			
1	1.4278082 + 2.1057120i			7.0120055E-15	2.9124502	5
2	1.4278082 + 2.1057120i	0.0000000	0.0000000	3.9252311E-16	2.9124502	20
3	2.5317830 + 2.1153793i	1.1039748	0.0096673	1.7554167E-16	3.0975499	6
4	5.1805941 + 2.1281372i	2.6488111	0.0127579	4.4464397E-15	2.8926347	32
5	5.1805941 + 2.1281372i	0.0000000	0.0000000	1.4443571E-15	2.8926347	19
6	5.1805941 + 2.1281372i	0.0000000	0.0000000	1.5313433E-15	2.8926347	20
7	7.6369533 + 2.1324590i	2.4563592	0.0043218	6.6813782E-14	2.7895566	8
8	8.4428388 + 2.1332475i	0.8058855	0.0007885	2.7950486E-13	2.7700081	7
9	9.2451439 + 2.1338619i	0.8023051	0.0006144	2.4184011E-15	2.7546587	7
10	10.0447401 + 2.1343492i	0.7995962	0.0004873	5.8291372E-12	2.7424194	6
11	10.8422355 + 2.1347418i	0.7974953	0.0003926	4.2525102E-14	2.7325215	6
12	11.6380675 + 2.1350624i	0.7958320	0.0003206	8.4686919E-15	2.7244146	6
13	12.4325594 + 2.1353274i	0.7944919	0.0002650	5.8054288E-15	2.7176978	6
14	13.2259554 + 2.1355489i	0.7933960	0.0002215	2.8296274E-10	2.7120746	5
15	14.0184435 + 2.1357358i	0.7924881	0.0001869	4.3085185E-11	2.7073226	5
16	14.8101707 + 2.1358950i	0.7917272	0.0001591	7.2837630E-12	2.7032725	5
17	15.6012539 + 2.1360315i	0.7910832	0.0001365	1.3510469E-12	2.6997938	5
18	16.3917870 + 2.1361495i	0.7905331	0.0001180	2.6768488E-13	2.6967848	5
19	17.1818465 + 2.1362522i	0.7900595	0.0001027	6.9014762E-14	2.6941650	5
20	17.1818465 + 2.1362522i	0.0000000	0.0000000	5.9369305E-13	2.6941650	41
21	17.9714953 + 2.1363421i	0.7896488	0.0000899	1.8169374E-14	2.6918707	5
22	18.7607855 + 2.1364212i	0.7892902	0.0000791	3.3003542E-15	2.6898503	5
23	19.5497608 + 2.1364912i	0.7889754	0.0000700	3.8548288E-15	2.6880622	5
24	20.3384582 + 2.1365534i	0.7886973	0.0000622	4.3058059E-15	2.6864723	5
25	21.1269087 + 2.1366089i	0.7884506	0.0000555	3.7975470E-15	2.6850525	5

Table 2: Eigenvalues ordered by increasing real part with initial guesses of the first type (3.21).

#	Initial Guess	Root	$\lvert F(z)\rvert$	$\lvert F'(z)\rvert$	Iter.
1	1.2244260 + 2.1373265i	1.4278082 + 2.1057120i	2.2734398E-11	2.9124502	5
2	2.2927663 + 2.1373265i	2.5317830 + 2.1153793i	2.0014830E-15	3.0975499	7
3	3.2479297 + 2.1373265i	3.4635004 + 2.1214556i	2.5852944E-15	3.0278346	7
4	4.1465047 + 2.1373265i	4.3363476 + 2.1254322i	9.2263808E-16	2.9520387	7
5	5.0127435 + 2.1373265i	5.1805941 + 2.1281372i	5.3707358E-13	2.8926347	6
6	5.8587721 + 2.1373265i	6.0085237 + 2.1300425i	1.1443917E-16	2.8481808	6
7	6.6913273 + 2.1373265i	6.8261900 + 2.1314265i	4.3088240E-16	2.8148661	6
8	7.5144511 + 2.1373265i	7.6369533 + 2.1324590i	6.1144639E-11	2.7895566	5
9	8.3307157 + 2.1373265i	8.4428388 + 2.1332475i	3.9459798E-12	2.7700081	5
10	9.1418358 + 2.1373265i	9.2451439 + 2.1338619i	3.0656935E-13	2.7546587	5
11	9.9489988 + 2.1373265i	10.0447401 + 2.1343492i	2.8038521E-14	2.7424194	5
12	10.7530525 + 2.1373265i	10.8422355 + 2.1347418i	5.2954991E-15	2.7325215	5
13	11.5546188 + 2.1373265i	11.6380675 + 2.1350624i	8.4686919E-15	2.7244146	5
14	12.3541640 + 2.1373265i	12.4325594 + 2.1353274i	5.8054288E-15	2.7176978	5
15	13.1520449 + 2.1373265i	13.2259554 + 2.1355489i	1.4804419E-15	2.7120746	5
16	13.9485389 + 2.1373265i	14.0184435 + 2.1357358i	2.2727393E-15	2.7073226	5
17	14.7438648 + 2.1373265i	14.8101707 + 2.1358950i	5.9468385E-16	2.7032725	5
18	15.5381980 + 2.1373265i	15.6012539 + 2.1360315i	1.3330083E-15	2.6997938	5
19	16.3316802 + 2.1373265i	16.3917870 + 2.1361495i	8.3624875E-11	2.6967848	4
20	17.1244275 + 2.1373265i	17.1818465 + 2.1362522i	4.0236877E-11	2.6941650	4
21	17.9165358 + 2.1373265i	17.9714953 + 2.1363421i	1.9972799E-11	2.6918707	4
22	18.7080849 + 2.1373265i	18.7607855 + 2.1364212i	1.0201163E-11	2.6898503	4
23	19.4991419 + 2.1373265i	19.5497608 + 2.1364912i	5.3547088E-12	2.6880622	4
24	20.2897636 + 2.1373265i	20.3384582 + 2.1365534i	2.8807934E-12	2.6864723	4
25	21.0799984 + 2.1373265i	21.1269087 + 2.1366089i	1.5871948E-12	2.6850525	4
26	21.8698877 + 2.1373265i	21.9151393 + 2.1366587i	8.9094205E-13	2.6837793	4
27	22.6594672 + 2.1373265i	22.7031730 + 2.1367035i	5.1239019E-13	2.6826335	4
28	23.4487681 + 2.1373265i	23.4910295 + 2.1367439i	2.9709074E-13	2.6815986	4
29	24.2378171 + 2.1373265i	24.2787263 + 2.1367805i	1.7984070E-13	2.6806608	4
30	25.0266379 + 2.1373265i	25.0662785 + 2.1368138i	1.0614944E-13	2.6798084	4
31	25.8152513 + 2.1373265i	25.8536992 + 2.1368442i	5.9422309E-14	2.6790313	4
32	26.6036756 + 2.1373265i	26.6410003 + 2.1368719i	3.7862604E-14	2.6783210	4
33	27.3919270 + 2.1373265i	27.4281920 + 2.1368974i	2.6775408E-14	2.6776700	4
34	28.1800198 + 2.1373265i	28.2152835 + 2.1369207i	1.3877928E-14	2.6770719	4
35	28.9679670 + 2.1373265i	29.0022830 + 2.1369422i	9.3905265E-15	2.6765212	4
36	29.7557801 + 2.1373265i	29.7891978 + 2.1369620i	1.0632383E-14	2.6760130	4
37	30.5434693 + 2.1373265i	30.5760345 + 2.1369804i	9.5155950E-15	2.6755430	4
38	31.3310439 + 2.1373265i	31.3627990 + 2.1369974i	2.7979686E-15	2.6751075	4
39	32.1185124 + 2.1373265i	32.1494965 + 2.1370132i	1.9363111E-14	2.6747032	4
40	32.9058823 + 2.1373265i	32.9361319 + 2.1370278i	2.0201628E-15	2.6743273	4

Table 3: First forty approximations with initial guesses of the second type (3.22).

#	Initial Guess	Root	$\|F(z)\|$	$\|F'(z)\|$	Iter.
1	0.0000000 + 0.0100000i	0.0000000 + 0.0307143i	5.5511151E-17	50.7193434	6
2	0.0000000 + 0.5000000i	0.0000000 + 0.2973021i	2.4825342E-16	14.5415019	14
3	0.0000000 + 1.1000000i	0.0000000 + 1.0264700i	1.1503888E-13	5.1047176	4
4	0.0000000 + 3.2500000i	0.0000000 + 3.1530080i	3.5566147E-15	5.2807041	5
5	0.0000000 + 3.8000000i	0.0000000 + 3.8278307i	1.9119140E-15	17.3759551	5

Table 4: Imaginary eigenvalues

#	Root	Difference		$\lvert F(z)\rvert$	$\lvert F'(z)\rvert$	Iter.
		Real	Complex			
1	1.4278082 + 2.1057120i			3.9252311E-16	2.9124502	9
2	2.5317830 + 2.1153793i	1.1039748	0.0096673	1.2412671E-16	3.0975499	6
3	3.4635004 + 2.1214556i	0.9317174	0.0060763	1.8554763E-15	3.0278346	6
4	4.3363476 + 2.1254322i	0.8728471	0.0039766	3.5373136E-13	2.9520387	6
5	5.1805941 + 2.1281372i	0.8442465	0.0027050	1.4443571E-15	2.8926347	8
6	6.0085237 + 2.1300425i	0.8279296	0.0019053	1.1443917E-16	2.8481808	6
7	6.8261900 + 2.1314265i	0.8176663	0.0013840	9.5544856E-13	2.8148661	5
8	7.6369533 + 2.1324590i	0.8107633	0.0010325	1.8273964E-14	2.7895566	6
9	8.4428388 + 2.1332475i	0.8058855	0.0007885	3.0028660E-15	2.7700081	9
10	9.2451439 + 2.1338619i	0.8023051	0.0006144	2.7254305E-13	2.7546587	6

Table 5: First ten complex eigenvalues.

#	n=100	n=200	n=300	$\|F(z)\|$
1	0.0000000+0.0307143i	0.0000000+0.0307143i	0.0000000+0.0307143i	1.1672571E-10
2	0.0000000+0.2973020i	0.0000000+0.2973020i	0.0000000+0.2973020i	1.2103375E-12
3	0.0000000+1.0264699i	0.0000000+1.0264699i	0.0000000+1.0264699i	7.1767207E-12
4	0.0000000+3.1530079i	0.0000000+3.1530079i	0.0000000+3.1530079i	1.5658855E-11
5	0.0000000+3.8278307i	0.0000000+3.8278307i	0.0000000+3.8278307i	2.7910373E-12
6*	0.0000000+7591.3520i	0.0000000+30049.769i	0.0000000+67381.310i	3.33E-01*
7	1.4278082+2.1057119i	1.4278082+2.1057119i	1.4278082+2.1057119i	4.7379817E-12
8	2.5317829+2.1153792i	2.5317829+2.1153792i	2.5317829+2.1153792i	6.9788742E-12
9	3.4635004+2.1214555i	3.4635004+2.1214555i	3.4635004+2.1214555i	4.6545777E-13
10	4.3363475+2.1254322i	4.3363475+2.1254322i	4.3363475+2.1254322i	3.6655158E-12
91	1.2207933+3.4347085i	68.305241+2.1372566i	68.305241+2.1372566i	4.7834858E-12
92	1.3202494+3.7912594i	69.090916+2.1372582i	69.090916+2.1372582i	5.5952315E-12
93	1.4266220+4.2946691i	69.876586+2.1372597i	69.876586+2.1372597i	7.1424803E-12
94	1.5525042+4.5172056i	70.662249+2.1372612i	70.662249+2.1372612i	9.8463479E-12
95	1.7218922+4.8822657i	71.447907+2.1372626i	71.447907+2.1372626i	3.6269868E-12
96	1.9338299+5.9112207i	72.233558+2.1372640i	72.233558+2.1372640i	9.9354170E-12
97	217.23044+7.5417016i	73.019205+2.1372653i	73.019205+2.1372653i	6.7034019E-12
98	248.01487+8.5195590i	73.804846+2.1372666i	73.804846+2.1372666i	5.6557245E-12
99	296.11072+10.439355i	74.590482+2.1372679i	74.590482+2.1372679i	4.3976431E-12
100	367.40755+16.140107i	75.376113+2.1372691i	75.376113+2.1372691i	6.7167333E-12

Table 6: Eigenvalues from 1 to 10 and from 91 to 100

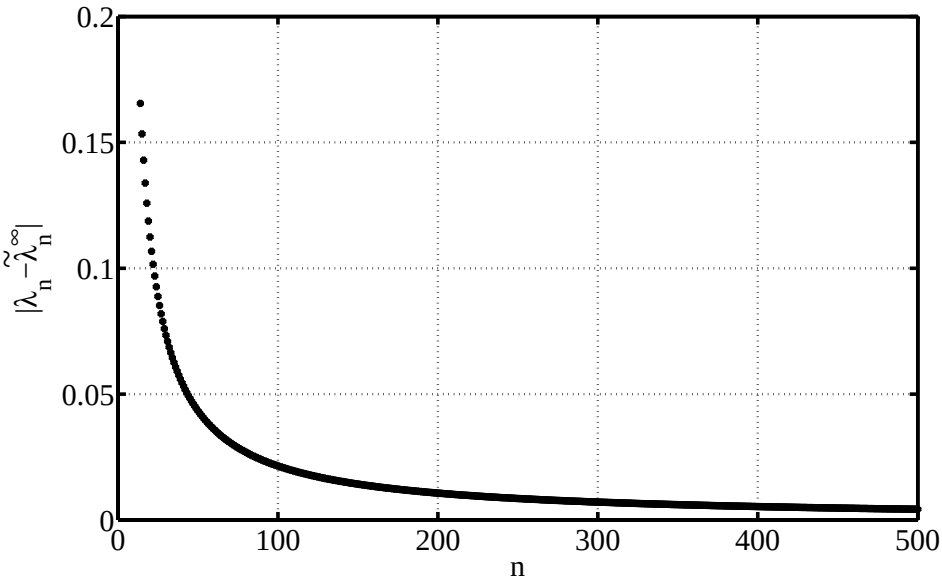

Figure 1: Distance of eigenvalues from the asymptotic values, with initial guesses of the first type (3.21).

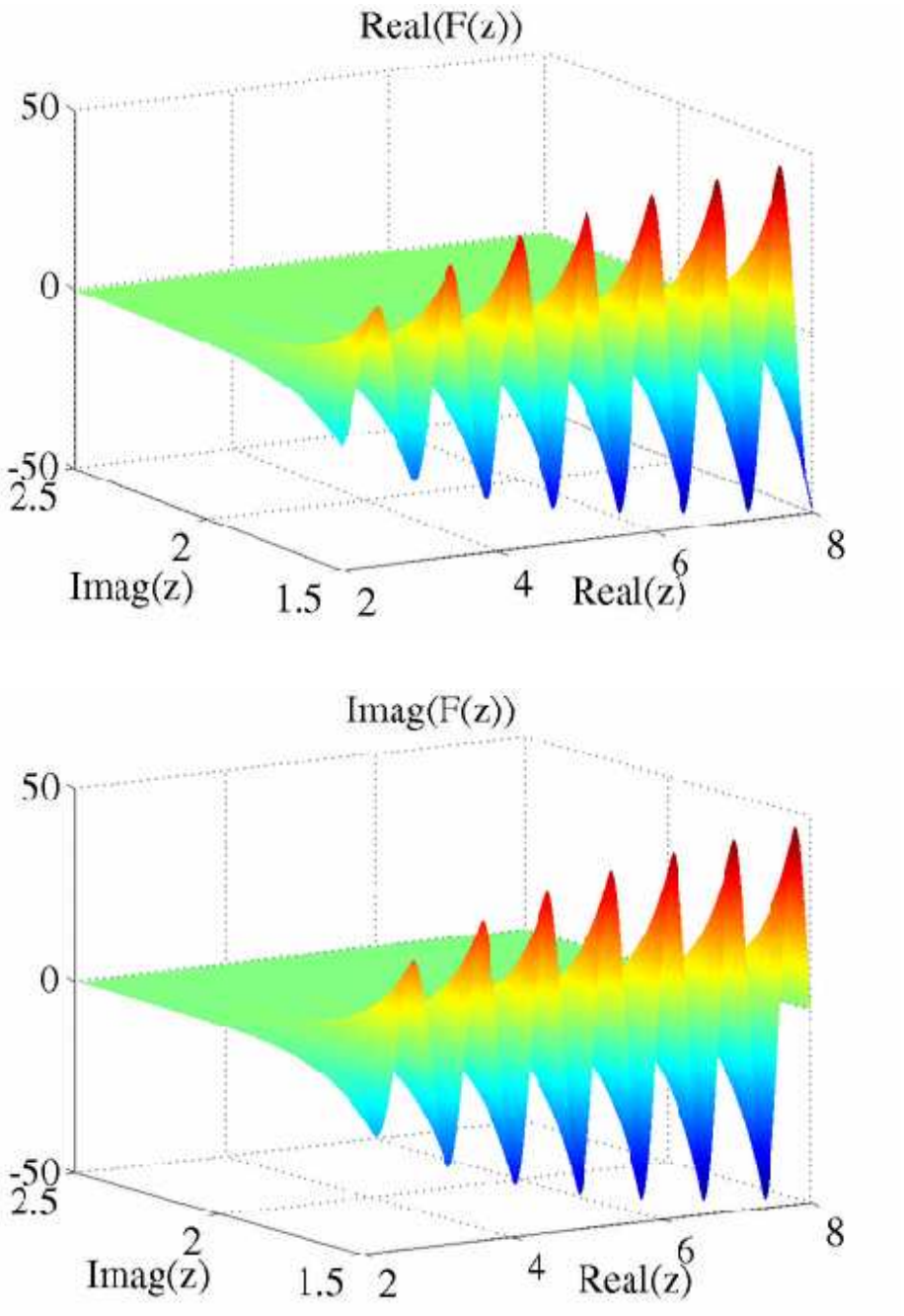

Figure 2: Real and complex part of the characteristic function.

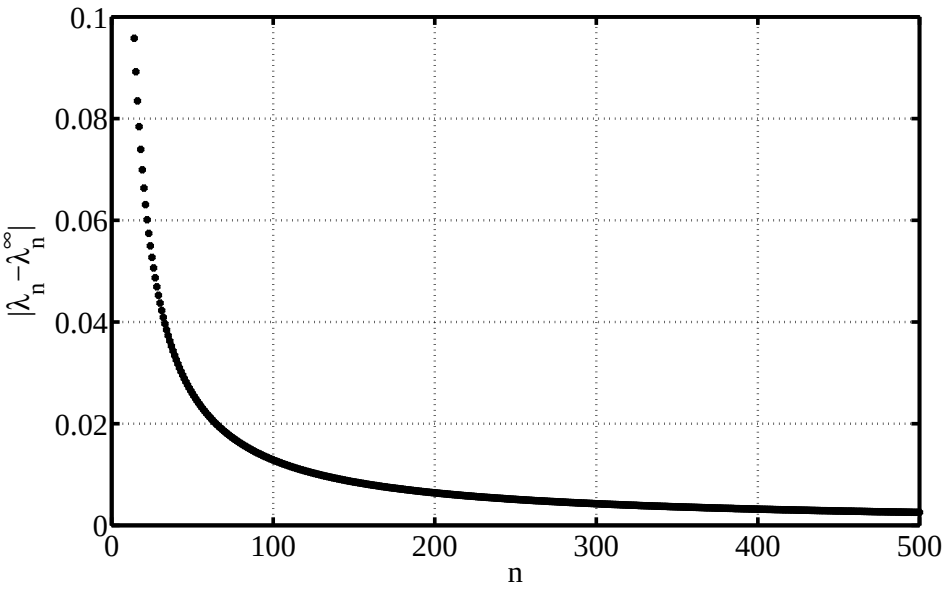

Figure 3: Distance of eigenvalues from the asymptotic values, with initial guesses of the second type (3.22).

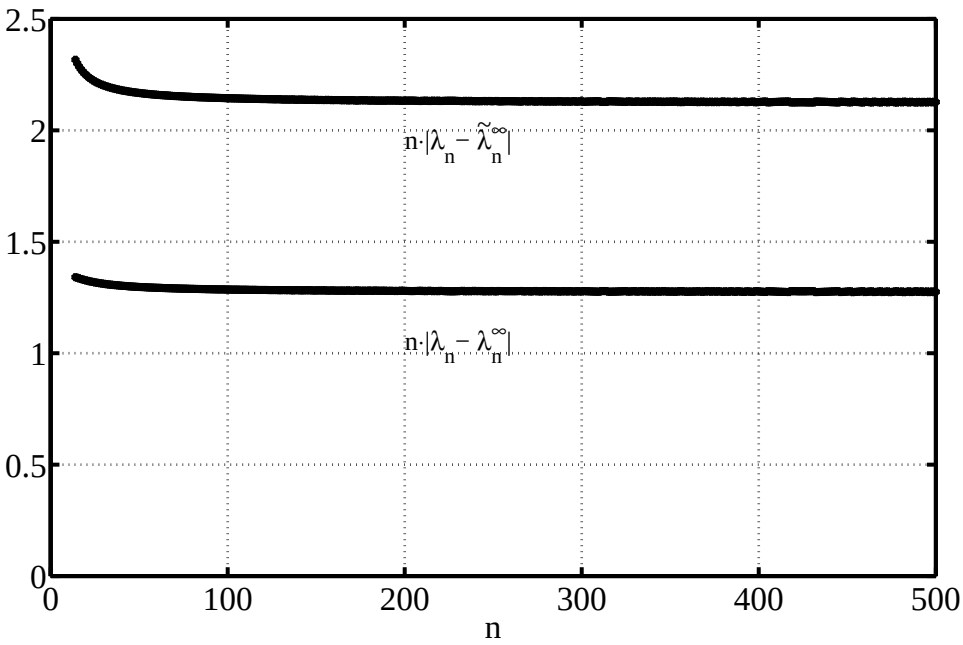

Figure 4: Comparison of convergence to approximate asymptotic values.

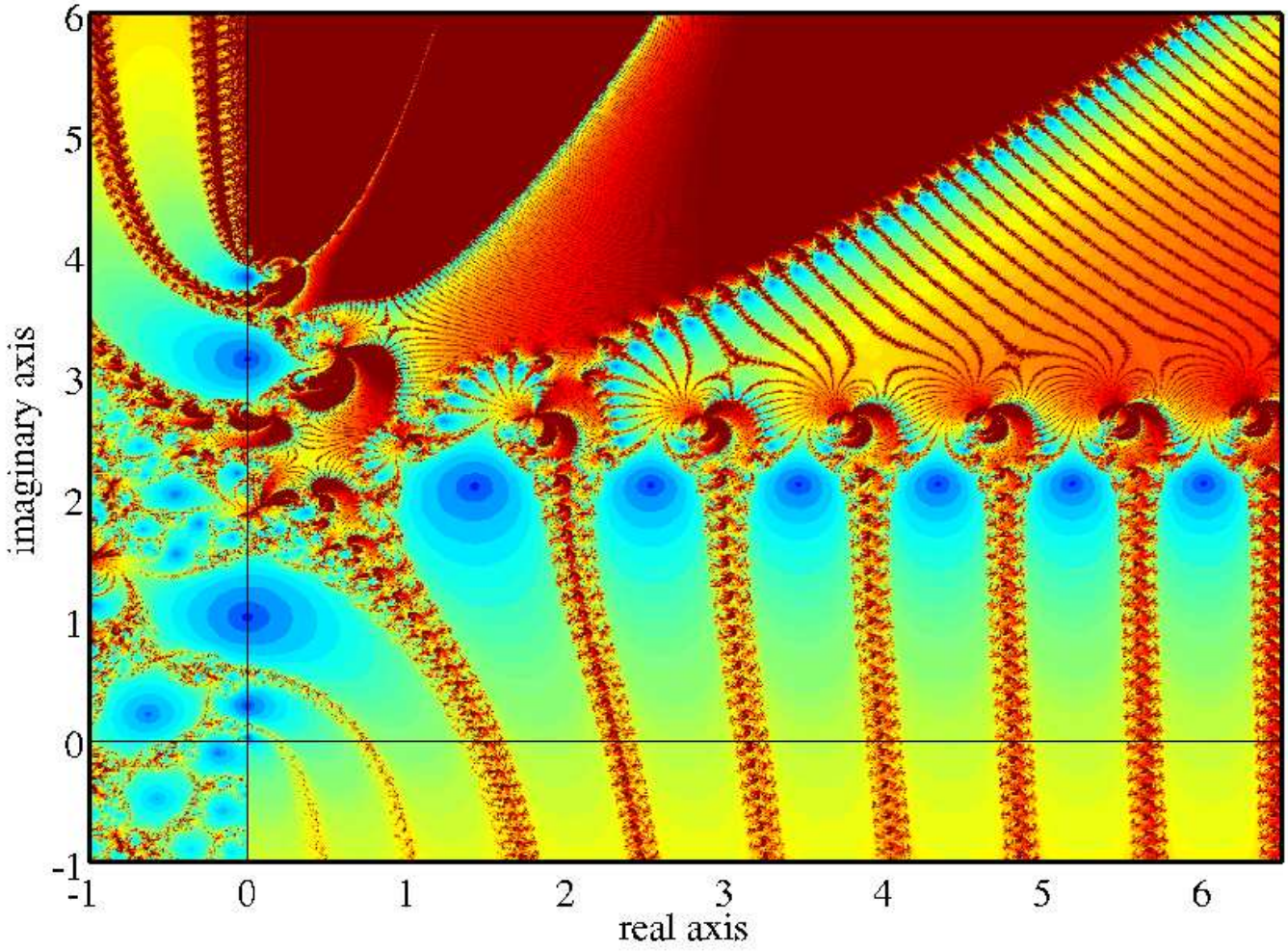

Figure 5: Color fractal image of convergence.

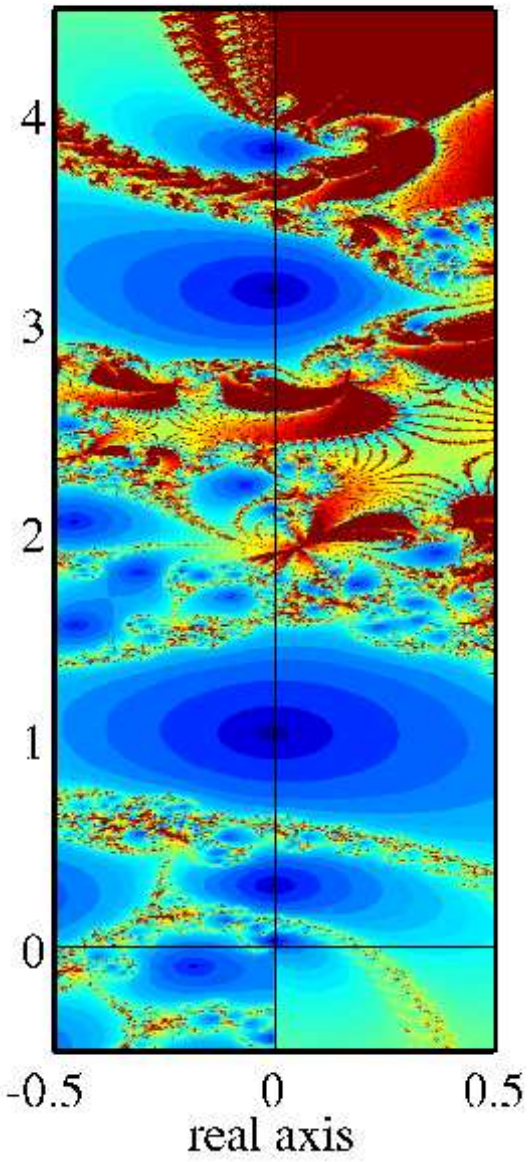

Figure 6: Imaginary eigenvalues.