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Automatica 44 (2008) 63–74
www.elsevier.com/locate/automatica

Output-feedback stabilization of an unstable wave equation ☆

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Received 7 September 2006; received in revised form 11 April 2007; accepted 11 May 2007
Available online 28 August 2007

Abstract

We consider the problem of stabilization of a one-dimensional wave equation that contains instability at its free end and control on the opposite end. In contrast to classical collocated "boundary damper" feedbacks for the neutrally stable wave equations with one end satisfying a homogeneous boundary condition, the controllers and the associated observers designed in the paper are more complex due to the open-loop instability of the plant. The controller and observer gains are designed using the method of "backstepping," which results in explicit formulae for the gain functions. We prove exponential stability and the existence and uniqueness of classical solutions for the closed-loop system. We also derive the explicit compensators in frequency domain. The results are illustrated with simulations.

Keywords: Distributed parameter systems; Wave equation; Backstepping; Stabilization; Boundary control

1. Introduction

Strings and flexible beams have been important benchmarks for the development of distributed parameter system theory for several decades. Even in the absence of damping (viscous, structural, or Kelvin–Voigt/material damping), when one end of a string/beam is *pinned/clamped*, they are neutrally stable (in the Lyapunov sense, i.e., in the sense that the overall energy is not increasing). They have another important property—that when actuation and sensing is performed on the same boundary (collocated boundary control), their input–output operator is passive or positive-real. The same operator is also *zero-state-observable* (in the sense of the definition in Khalil (2001), which is provable by a LaSalle-type argument). Due to passivity and zero-state-observability, strings and beams with one end pinned/clamped are stabilizable by proportional feedback (or by derivative feedback, depending on exactly which physical quantity at the boundary is considered to be the actuated input and the measured output).

As a result of stabilizability by simple feedback laws, collocated boundary control systems have been under study for several decades and have spawned hundreds of papers for strings, beams, and thin plates. Various system-theoretic aspects such as well posedness, stabilizability, and detectability of strings, cables, and beams were considered by Lions (1988), Chen (1979), Lagnese (1983), Komornik and Zuazua (1990), Weiss and Curtain (1997), Bardos, Lebeau, and Rauch (1992), Bardos, Halpern, Lebeau, Rauch, and Zuazua (1991), Tucsnak (1993), Rebarber and Zwart (1998). The observer design was considered, among others, by Demetriou (2004), and actuator and sensor positioning and design was addressed by Fahroo and Demetriou (2000). The book by Luo, Guo, and Morgul (1998) is dedicated to collocated boundary control for beams, whereas the book by Lagnese (1989) is dedicated to collocated boundary control of thin plates. The book by Lasiecka (2002) carries the collocated concept further to coupled structural-acoustic nonlinear models. As many authors acknowledge, collocated feedback laws are primarily to be understood as being implementable using *passive control* devices, such as

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Faryar Jabbari under the direction of Editor Roberto Tempo.

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doi:10.1016/j.automatica.2007.05.012

"dampers." However, active control implementations have been pursued using smart material actuators/sensors (PZT, PVDF) by Moheimani and Fleming (2006), Banks, Smith, Brown, Silcox, and Metcalf (1997), Preumont, De Marneffe, Deraemaeker, and Bossens (2005), Sodano, Inman, and Belvin (2006), or using air jets and electromagnets by de Queiroz, Dawson, Agarwal, and Zhang (1999), de Queiroz and Rahn (2002) and Zhang, Dawson, de Queiroz, and Vedagarbha (1997).

When the actuator and sensor are not collocated, the exponential stabilization problem changes dramatically. The input–output operator is typically no longer passive, which is the case for several reasons—either because the relative degree changes (in the collocated case it is no greater than one) or because the system transfer function becomes "non-minimum phase"—which precludes the application of simple P/PD/PI controllers. It is clear that one should expect that, in general, non-collocated actuator/sensor pairs require higher order dynamic compensator, including possibly compensators arrived at using observer-based control designs.

Of particular interest in many applications are "anti-collocated" architectures, where the actuator acts through one boundary of the structure (beam, string, etc.), whereas the sensor is placed on the other boundary. Such problems were solved for one-dimensional parabolic PDEs by Smyshlyaev and Krstic (2005) who considered unstable reaction–advection–diffusion equations and designed output-feedback controllers by the method of "backstepping," which uses the Volterra transformation to map an unstable PDE into a stable "target" PDE. These controllers employed state feedback laws applied through one boundary condition, where the state estimate was supplied by an observer PDE driven by a measurement at the other boundary of the PDE domain. Until recently, the applicability of the backstepping method was restricted to parabolic PDEs. The main difficulty in extending this method to flexible structures lied in the fact that one cannot add in-domain damping (neither viscous nor Kelvin–Voigt) to a hyperbolic PDE using Volterra transformation (while it is possible to do that for the heat equation). Therefore the right "target" system in the hyperbolic case is not obvious at all. In the conference paper, Krstic, Smyshlyaev, and Siranosian (2006) extended the backstepping method to wave equations and Timoshenko beam models with Kelvin–Voigt damping. The present paper expands upon the ideas in Krstic et al. (2006) for the undamped wave equation with two actuator/sensor configurations.

Guo and Xu (2007) considered a wave equation with an anti-collocated actuator/sensor configuration, where one end of the string was pinned and Neumann measurement was performed at that end, whereas the other end was actuated using Neumann actuation. Fig. 1 shows the actuator/sensor configuration in Guo and Xu (2007) (bottom), in comparison with the configuration used in the classical "boundary damper" control approach (top). The authors in Guo and Xu (2007) proposed an observer-based compensator which exponentially stabilizes the string. The stability is analyzed using the Riesz basis approach. The observer in Guo and Xu (2007) is designed using an approach *dual* to the approach that exploits the passivity of the collocated configuration and leads to PD control. To be exact, the output injection operator of the observer (the observer gain operator) is chosen as the adjoint of the measurement operator (times a constant). In physical terms, this means that the output injection is applied only in the boundary condition of the observer (rather than also in the domain, i.e., on the right-hand side of the PDE, as would be the case in a general observer design problem for a PDE). Hence, the overall output-feedback compensator consists of a simple controller and a simple observer. The problem considered in Guo and Xu (2007) is harder than the classical problem but it is still solvable using a simple controller and observer.

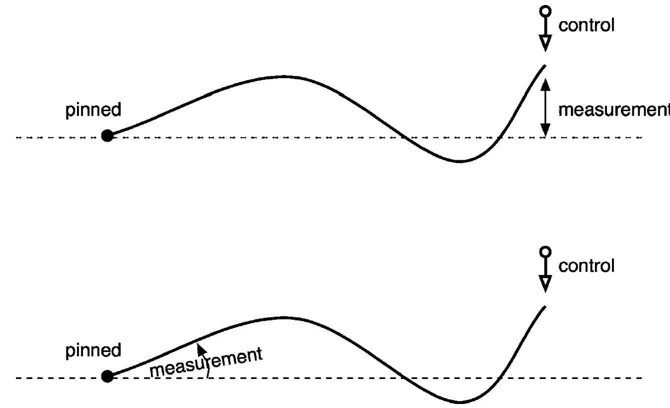

Fig. 1. Two problem formulations for boundary control of a *stable* string pinned on one end: top—classical formulation with the actuation/sensing collocated on the right end (Neuman/force actuation and Dirichlet/displacement sensing). Bottom (Guo & Xu, 2007)—Neumann actuation on the right end and Neumann sensing on the left (pinned) end.

In this paper we consider a much harder problem where, instead of having one end of the string pinned, we have it subjected to a destabilizing boundary condition. The exact form of the "destabilizing" boundary condition is introduced in Section 2, however, we point out here that the boundary condition is of Robin type, that it results in the uncontrolled system having some real positive eigenvalues, and that the physical cause of such a boundary condition can be the action of an "unfavorably" polarized magnetic field to a metallic free end of the string. As shown in Fig. 2, we allow ourselves to apply control only through the boundary condition on the end of the string opposite from where the destabilizing force acts, which is what makes the problem difficult.

We present two results in the paper. The first result, corresponding to the actuator/sensor configuration in the top picture in Fig. 2, is presented in Sections 2–5. This result employs a non-trivial backstepping control law, whereas the observer design is straightforward. Then, in Section 6 we present a control design for the problem in the bottom picture in Fig. 2, which employs both actuation and sensing on the boundary opposite from the boundary with the destabilizing effects. This configuration results in both the controller and the observer requiring a non-trivial choice for both the controller and the observer, where the gain functions are derived using the backstepping approach. In particular, the observer in Section 6 must incorporate the output injection not in the boundary condition where

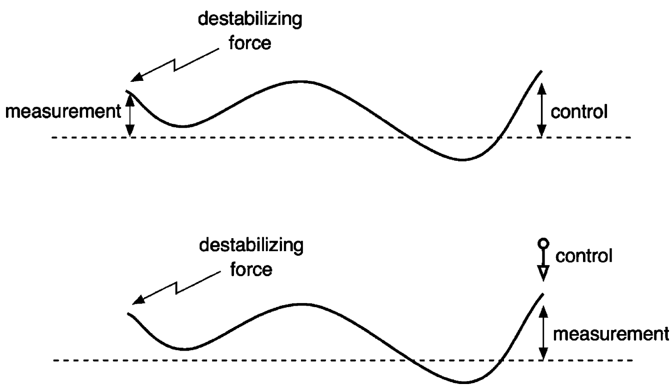

Fig. 2. Two problem formulations for boundary control of an *unstable* string with a destabilizing force (boundary condition) on the free end: top (Sections 2–5)—Dirichlet actuation on the right end, Dirichlet sensing on the left end. Bottom (Section 6)—collocated actuation/sensing on the right end (Neumann/force actuation and Dirichlet/displacement sensing).

the measurement is taken but on the right-hand side of the PDE, throughout the domain.

While the backstepping method does not provide optimality, the controller and the observer gain functions are obtained in closed form as functions of the spatial variable x (this dependence happens to be exponential) and the parameters of the plant, which is not the case for the optimality-driven approaches (where operator Riccati equations have to be solved numerically).

We should mention that the time derivative of the output is used in our design, under the assumption of a noise-free output measurement. This is not a problem particular to our design, but a common issue in all the existing approaches to boundary control of flexible structures.

The paper is organized as follows. In Sections 2–5 we present the problem formulation, controller/observer design, stability analysis, and well posedness analysis for our anti-collocated Dirichlet actuation/sensing problem. Then, in Section 6, given that the main techniques are covered in the previous sections, we briefly present the design and the analysis for our collocated case (with a destabilizing b.c.). Frequency domain representations of the compensators are derived in Section 7. In Section 8 we present simulation results.

2. A non-collocated problem

We consider the system:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad 0<x<1,\ t>0, \\ w_x(0,t) = -qw(0,t), \quad t \geqslant 0, \\ w(1,t) = u(t), \quad t\geqslant 0, \\ w(x,0) = w_0(x), \quad 0<x<1, \\ y(t) = w(0,t), \quad t\geqslant 0, \end{cases} \tag{1}$$

where u is a (scalar) control input and y is a (scalar) measured output. The objective is to exponentially stabilize the system to zero in energy state space. We present a dynamic compensator which employs a PDE observer and full state feedback based on the observer state.

For $q=0$, Eq. (1) models a string which is free at the end $x=0$ and actuated by displacement actuation at the end $x=1$. For $q\neq 0$ the free end of the string is subject to a force proportional to the displacement, which physically may be the result of various phenomena. For example, if the $x=0$ end of the string is made of iron and it is placed between two magnets of the same polarity, the string's end will be subject to a magnetic force which depends on its displacement (the magnetic force will typically depend on the displacement nonlinearly, however, for small signals this force can be approximated using linear dependence). When $q>0$ the zero equilibrium state of the system will become unstable. Physically this would correspond to having magnets whose polarity is such that they both generate an attractive force.

3. Controller and observer design

We design the following observer for system (1):

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \quad 0<x<1, \quad t>0, \\ \widehat{w}_x(0,t) = -qy(t) - c_0[\dot{y}(t) - \widehat{w}_t(0,t)], \quad t\geqslant 0, \\ \widehat{w}(1,t) = u(t), \quad t\geqslant 0, \\ \widehat{w}(x,0) = \widehat{w}_0(x), \quad 0<x<1, \end{cases} \tag{2}$$

where c_0 is a positive design parameter. The observer (2) is a "natural observer" (Demetriou, 2004) in a sense that it employs a copy of the plant plus output injection (in this case, only at the boundary).

To show the exponential convergence of the observer above, let $\varepsilon(x,t) = w(x,t) - \widehat{w}(x,t)$ denote the observer error. Then, it is easy to see that ε is governed by

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t), \quad 0<x<1, \quad t>0, \\ \varepsilon_x(0,t) = c_0\varepsilon_t(0,t), \quad t\geqslant 0, \\ \varepsilon(1,t) = 0, \quad t\geqslant 0, \\ \varepsilon(x,0) = \varepsilon_0(x), \quad 0<x<1. \end{cases} \tag{3}$$

System (3) is the familiar damped wave equation, with the greatest damping obtained for $c_0 = 1$. The initial condition $\varepsilon_0(x)$ is, of course, not necessarily equal to zero since the initial condition of the observer $\widehat{w}_0(x)$ can be chosen arbitrarily.

We propose the following observer-based feedback controller (the motivation behind it will be clear from (6) to (9)):

$$u(t) = \widehat{w}(1,t),$$

$$\begin{aligned} \widehat{w}_x(1,t) = & -(c_1+q)\int_0^1 e^{q(1-\xi)}[c_2\widehat{w}_t(\xi,t) + q\widehat{w}(\xi,t)]\,d\xi \\ & - c_2\widehat{w}_t(1,t) - (c_1+q)\widehat{w}(1,t), \end{aligned} \tag{4}$$

where c_1, c_2 are positive design parameters. Then (2) becomes[1]

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \\ \widehat{w}_x(0,t) = -qy(t) - c_0[\dot{y}(t) - \widehat{w}_t(0,t)], \\ \widehat{w}_x(1,t) = -c_2\widehat{w}_t(1,t) - (c_1+q)\widehat{w}(1,t) \\ \qquad -(c_1+q)\int_0^1 e^{q(1-\xi)}[c_2\widehat{w}_t(\xi,t)+q\widehat{w}(\xi,t)]\,d\xi. \end{cases} \tag{5}$$

[1] In the rest of the paper we do not explicitly state the initial conditions for the PDEs. We also omit the (obvious) intervals for t and x ($t\geqslant 0$ and $0<x<1$).

It should be noted that the control $u(t)$ is implemented based on the boundary value $\widehat{w}(1,t)$ of the solution $\widehat{w}(x,t)$ of the observer system. Thus, it is better to think of (4) as a boundary condition to the observer system rather than an *implicit* expression for the control law, because the control $u(t)=\widehat{w}(1,t)$ is really given explicitly in terms of the observer state. The recommended choices of the control gains are c_2 around one, and c_1 relatively large.

While the motivation behind the observer (2) is simple, the motivation behind the controller design (4) is a little more intricate. The choice of the boundary control (4) comes from the backstepping design (Smyshlyaev & Krstic, 2004). Consider the invertible change of variable

$$\begin{aligned}\widetilde{w}(x,t)&=[(I+\mathbb{P})\widehat{w}](x,t)\\&=\widehat{w}(x,t)+(c_1+q)\int_0^x \mathrm{e}^{q(x-\xi)}\widehat{w}(\xi,t)\,\mathrm{d}\xi,\end{aligned}\tag{6}$$

where $\mathbb{P}$ is a Volterra transformation. The most important properties of the Volterra transformation that are crucial for the designs presented here are boundedness, invertibility, and (spatial) causality. In this paper all the Volterra transformations and their inverses are given explicitly, which makes these properties almost obvious. For general theory of Volterra transformations, see Gripenberg, Londen, and Staffans (1990). The inverse of $(I+\mathbb{P})$ is given by

$$\begin{aligned}\widehat{w}(x,t)&=[(I+\mathbb{P})^{-1}\widetilde{w}](x,t)\\&=\widetilde{w}(x,t)-(c_1+q)\int_0^x \mathrm{e}^{-c_1(x-\xi)}\widetilde{w}(\xi,t)\,\mathrm{d}\xi.\end{aligned}\tag{7}$$

It can be shown that (6) converts the system (5) into

$$\begin{cases}\widetilde{w}_{tt}(x,t)=\widetilde{w}_{xx}(x,t)\\ \qquad\qquad +(c_1+q)\mathrm{e}^{qx}[q\varepsilon(0,t)+c_0\varepsilon_t(0,t)],\\ \widetilde{w}_x(0,t)=c_1\widetilde{w}(0,t)-[q\varepsilon(0,t)+c_0\varepsilon_t(0,t)],\\ \widetilde{w}_x(1,t)=-c_2\widetilde{w}_t(1,t).\end{cases}\tag{8}$$

Thus, the overall system is a cascade of the exponentially stable $\varepsilon(x,t)$-subsystem and the $\widetilde{w}(x,t)$-subsystem, as given in (8). For $\varepsilon(0,t)\equiv 0$, the resulting system (8) is exponentially stable[2] :

$$\begin{cases}\widetilde{w}_{tt}(x,t)=\widetilde{w}_{xx}(x,t),\\ \widetilde{w}_x(0,t)=c_1\widetilde{w}(0,t),\\ \widetilde{w}_x(1,t)=-c_2\widetilde{w}_t(1,t).\end{cases}\tag{9}$$

This is a familiar form of a wave equation with a "passive damper" boundary condition at $x=1$, except that at $x=0$ we have a mixed boundary condition rather than the Dirichlet one. However, for large values of c_1 this boundary condition is "almost" Dirichlet which, together with c_2 being close to 1, makes the system exponentially stable. The idea of the transformation (6) is that it makes the closed-loop system (5) behave as the system (9) (in the absence of observer) by propagating the destabilizing q-term from the boundary $x=0$, through the entire domain, to the boundary $x=1$, where it gets cancelled by the feedback.

[2] In the appropriate norms, the exact topology will be defined later.

The stability of the overall cascade of ε and $\widetilde{w}$ systems, for $\varepsilon(x,t)\neq 0$, will be shown in the next section using a Lyapunov method (that proof will essentially be a proof of the separation principle for our observer and state-feedback controller). Since the transformed system is related to the original one via the invertible transformation (6), the original system with the output feedback controller is also exponentially stable in the appropriate norms (Section 5).

4. Well-posedness and stability of transformed system

In this section, we consider the overall system (3), (8):

$$\begin{cases}\varepsilon_{tt}(x,t)=\varepsilon_{xx}(x,t),\\ \varepsilon_x(0,t)=c_0\varepsilon_t(0,t),\quad \varepsilon(1,t)=0,\\ \widetilde{w}_{tt}(x,t)=\widetilde{w}_{xx}(x,t)\\ \qquad\qquad +(c_1+q)\mathrm{e}^{qx}[q\varepsilon(0,t)+c_0\varepsilon_t(0,t)],\\ \widetilde{w}_x(0,t)=c_1\widetilde{w}(0,t)-[q\varepsilon(0,t)+c_0\varepsilon_t(0,t)],\\ \widetilde{w}_x(1,t)=-c_2\widetilde{w}_t(1,t),\end{cases}\tag{10}$$

in the space $H=H_R^1(0,1)\times L^2(0,1)\times H^1(0,1)\times L^2(0,1)$, $H_R^1(0,1)=\{f\in H^1(0,1)|f(1)=0\}$ with the inner product

$$\begin{aligned}&\langle (f_1,g_1,\phi_1,\psi_1),(f_2,g_2,\phi_2,\psi_2)\rangle\\ &\quad=c_1\phi_1(0)\overline{\phi_2(0)}+K\int_0^1[f_1'(x)\overline{f_2'(x)}+g_1(x)\overline{g_2(x)}]\,\mathrm{d}x\\ &\qquad+K\delta_0\int_0^1(x-2)[f_1'(x)\overline{g_2(x)}+g_1(x)\overline{f_2'(x)}]\,\mathrm{d}x\\ &\qquad+\int_0^1[\phi_1'(x)\overline{\phi_2'(x)}+\psi_1(x)\overline{\psi_2(x)}]\,\mathrm{d}x\\ &\qquad+\delta\int_0^1(x+1)[\phi_1'(x)\overline{\psi_2(x)}+\psi_1(x)\overline{\phi_2'(x)}]\,\mathrm{d}x\\ &\qquad\forall (f_1,g_1,\phi_1,\psi_1,f_2,g_2,\phi_2,\psi_2)\in H,\end{aligned}$$

where $0<\delta_0<\min\{\frac{1}{2},c_0/(1+c_0^2)\}$, $0<\delta<\min\{\frac{1}{2},c_2/(1+c_2^2)\}$ and $K>0$ is large enough so that A is dissipative in H as in the proof of Lemma 1. It is easy to see that the above inner product is well-defined. We only need to check the positivity since other properties are obvious. Indeed, for any $(f,g,\phi,\psi)\in H$ we have

$$\begin{aligned}\|(f,g,\phi,\psi)\|^2&\geqslant K(1-2\delta_0)\int_0^1[|f'(x)|^2+|g(x)|^2]\,\mathrm{d}x\\ &\quad+(1-2\delta)\int_0^1[|\phi'(x)|^2+|\psi(x)|^2]\,\mathrm{d}x.\end{aligned}$$

Define the operator $A:D(A)(\subset H)\to H$ as follows:

$$\begin{cases}A(f,g,\phi,\psi)\\ \quad=(g,f'',\psi,\phi''+(c_1+q)\mathrm{e}^{q\cdot}[qf(0)+c_0g(0)]),\\ D(A)=\{(f,g,\phi,\psi)\in(H^2(0,1)\cap H_R^1(0,1))\\ \qquad\times H_R^1(0,1)\times H^2(0,1)\times H^1(0,1)|,\\ f'(0)=c_0g(0),\ \phi'(0)=c_1\phi(0)-[qf(0)+c_0g(0)],\\ \phi'(1)=-c_2\psi(1)\},\quad \forall (f,g,\phi,\psi)\in D(A).\end{cases}\tag{11}$$

Then system (10) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\cdot,t)=A\mathbf{v}(\cdot,t) \tag{12}$$

where $\mathbf{v}=(\varepsilon,\varepsilon_t,\widetilde{w},\widetilde{w}_t)$.

Lemma 1. *Let A be defined by* (11). *Then A generates an exponentially stable C_0-semigroup on H. Therefore, for any initial value $\mathbf{v}(\cdot,0)\in H$, there exists a unique solution to* (12) *such that $\mathbf{v}(\cdot,t)\in C([0,\infty);H)$, and there are positive constants M,ω such that*

$$\|\mathbf{v}(\cdot,t)\|\leqslant M\mathrm{e}^{-\omega t}\|\mathbf{v}(\cdot,0)\|. \tag{13}$$

Moreover, if $\mathbf{v}(\cdot,0)\in D(A)$, then $\mathbf{v}(\cdot,t)\in C^1([0,\infty);H)$ is the classical solution of (10).

Proof. Define the Lyapunov functions:

$$E_\varepsilon(t)=\frac{1}{2}\int_0^1[\varepsilon_x^2(x,t)+\varepsilon_t^2(x,t)]\,\mathrm{d}x+\delta_0\int_0^1(-2+x)\varepsilon_x(x,t)\varepsilon_t(x,t)\,\mathrm{d}x, \tag{14}$$

and

$$E_{\widetilde{w}}(t)=\frac{1}{2}\int_0^1[\widetilde{w}_x^2(x,t)+\widetilde{w}_t^2(x,t)]\,\mathrm{d}x+\frac{c_1}{2}\widetilde{w}^2(0,t)+\delta\int_0^1(1+x)\widetilde{w}_x(x,t)\widetilde{w}_t(x,t)\,\mathrm{d}x. \tag{15}$$

Both of them are positive definite for $\delta_0<\frac{1}{2}$, $\delta<\frac{1}{2}$, as simple computation shows:

$$E_\varepsilon(t)\geqslant\left(\frac{1}{2}-\delta_0\right)\int_0^1[\varepsilon_x^2(x,t)+\varepsilon_t^2(x,t)]\,\mathrm{d}x, \tag{16}$$

$$E_{\widetilde{w}}(t)\geqslant\frac{c_1}{2}\widetilde{w}^2(0,t)+\left(\frac{1}{2}-\delta\right)\int_0^1[\widetilde{w}_x^2(x,t)+\widetilde{w}_t^2(x,t)]\,\mathrm{d}x. \tag{17}$$

The time derivatives of E_ε and $E_{\widetilde{w}}$ along the trajectory of (10) are, respectively,

$$\begin{aligned}\dot{E}_\varepsilon(t)=&\int_0^1[\varepsilon_x(x,t)\varepsilon_{xt}(x,t)+\varepsilon_t(x,t)\varepsilon_{xx}(x,t)]\,\mathrm{d}x\\&+\delta_0\int_0^1(-2+x)\varepsilon_{xt}(x,t)\varepsilon_t(x,t)\,\mathrm{d}x\\&+\delta_0\int_0^1(-2+x)\varepsilon_x(x,t)\varepsilon_{xx}(x,t)\,\mathrm{d}x\\=&\,\varepsilon_x(x,t)\varepsilon_t(x,t)|_0^1-\frac{\delta_0}{2}\int_0^1[\varepsilon_x^2(x,t)+\varepsilon_t^2(x,t)]\,\mathrm{d}x\\&+\frac{\delta_0}{2}(-2+x)(\varepsilon_t^2(x,t)+\varepsilon_x^2(x,t))|_0^1\\=&-\frac{\delta_0}{2}\int_0^1[\varepsilon_x^2(x,t)+\varepsilon_t^2(x,t)]\,\mathrm{d}x\\&-[c_0-\delta_0(1+c_0^2)]\varepsilon_t^2(0,t)-\frac{\delta_0}{2}\varepsilon_x^2(1,t),\end{aligned} \tag{18}$$

$$\begin{aligned}\dot{E}_{\widetilde{w}}(t)=&\int_0^1[\widetilde{w}_x(x,t)\widetilde{w}_{xt}(x,t)+\widetilde{w}_t(x,t)\widetilde{w}_{xx}(x,t)]\,\mathrm{d}x\\&+c_1\widetilde{w}(0,t)\widetilde{w}_t(0,t)\\&+\delta\int_0^1(1+x)\widetilde{w}_{xt}(x,t)\widetilde{w}_t(x,t)\,\mathrm{d}x\\&+\delta\int_0^1(1+x)\widetilde{w}_x(x,t)\widetilde{w}_{xx}(x,t)\,\mathrm{d}x\\=&-\frac{\delta}{2}\int_0^1[\widetilde{w}_x^2(x,t)+\widetilde{w}_t^2(x,t)]\,\mathrm{d}x-\frac{\delta}{2}\widetilde{w}_t^2(0,t)\\&-[c_2-\delta(1+c_2^2)]\widetilde{w}_t^2(1,t)\\&-\frac{\delta}{2}[c_1\widetilde{w}(0,t)-(q\varepsilon(0,t)+c_0\varepsilon_t(0,t))]^2\\&+[q\varepsilon(0,t)+c_0\varepsilon_t(0,t)]\Big\{\widetilde{w}_t(0,t)+(c_1+q)\\&\times\int_0^1\mathrm{e}^{qx}[\widetilde{w}_t(x,t)+\delta(1+x)\widetilde{w}_x(x,t)]\,\mathrm{d}x\Big\}.\end{aligned} \tag{19}$$

By performing two completions of squares further, we get

$$\begin{aligned}\dot{E}_{\widetilde{w}}(t)=&-\frac{\delta}{4}\int_0^1[\widetilde{w}_x^2(x,t)+\widetilde{w}_t^2(x,t)]\,\mathrm{d}x\\&-[c_2-\delta(1+c_2^2)]\widetilde{w}_t^2(1,t)\\&-\frac{\delta}{4}[\widetilde{w}_t^2(0,t)+c_1^2\widetilde{w}^2(0,t)]\\&-\frac{\delta}{4}\left[c_1\widetilde{w}(0,t)-\frac{2}{c_1}(q\varepsilon(0,t)+c_0\varepsilon_t(0,t))\right]^2\\&-\frac{\delta}{4}\left[\widetilde{w}_t(0,t)-\frac{2}{\delta}(q\varepsilon(0,t)+c_0\varepsilon_t(0,t))\right]^2\\&-\frac{\delta}{4}\int_0^1[\widetilde{w}_x(x,t)-2(c_1+q)(1+x)\\&\times\mathrm{e}^{qx}(q\varepsilon(0,t)+c_0\varepsilon_t(0,t))]^2\,\mathrm{d}x\\&-\frac{\delta}{4}\int_0^1\left[\widetilde{w}_t(x,t)-2\frac{c_1+q}{\delta}\mathrm{e}^{qx}(q\varepsilon(0,t)\right.\\&\left.+c_0\varepsilon_t(0,t))\right]^2\mathrm{d}x\\&+\left[(c_1+q)\int_0^1\left(\delta(1+x)^2+\frac{1}{\delta}\right)\mathrm{e}^{2qx}\,\mathrm{d}x\right.\\&\left.-\frac{1}{2}+\frac{1}{\delta}+\frac{\delta}{c_1^2}\right](q\varepsilon(0,t)+c_0\varepsilon_t(0,t))^2.\end{aligned} \tag{20}$$

With Poincaré inequality, $\varepsilon^2(0,t)\leqslant 4\|\varepsilon_x(\cdot,t)\|^2_{L^2(0,1)}$, we obtain

$$\begin{aligned}\dot{E}_{\widetilde{w}}(t)\leqslant&-\frac{\delta}{4}\int_0^1[\widetilde{w}_x^2(x,t)+\widetilde{w}_t^2(x,t)]\,\mathrm{d}x-\frac{\delta}{4}c_1^2\widetilde{w}^2(0,t)\\&-\frac{\delta}{4}\widetilde{w}_t^2(0,t)-[c_2-\delta(1+c_2^2)]\widetilde{w}_t^2(1,t)\\&+\Lambda\left[4q^2\int_0^1\varepsilon_x^2(x,t)\,\mathrm{d}x+c_0^2\varepsilon_t^2(0,t)\right],\end{aligned} \tag{21}$$

where

$$\Lambda = 2\left[\frac{1}{\delta} + \frac{\delta}{c_1^2} + \left(4\delta + \frac{1}{\delta}\right)(c_1+q)^2 e^{2q}\right]. \tag{22}$$

We now take the overall Lyapunov function as

$$E(t) = KE_\varepsilon(t) + E_{\tilde{w}}(t) \tag{23}$$

and compute its derivative along the solution of (10):

$$\begin{aligned}\dot{E}(t) \leqslant & -\frac{\delta}{4}\int_0^1 [\widetilde{w}_x^2(x,t) + \widetilde{w}_t^2(x,t)]\,dx - \frac{\delta}{4}c_1^2\widetilde{w}^2(0,t) \\ & -\frac{\delta}{4}\widetilde{w}_t^2(0,t) - \left[K\frac{\delta_0}{2} - 4q^2\Lambda\right]\int_0^1 \varepsilon_x^2(x,t)\,dx \\ & -(c_2 - \delta(1+c_2^2))\widetilde{w}_t^2(1,t) - K\frac{\delta_0}{2}\int_0^1 \varepsilon_t^2(x,t)\,dx \\ & -[K(c_0 - \delta_0(1+c_0^2)) - c_0^2\Lambda]\varepsilon_t^2(0,t),\end{aligned} \tag{24}$$

for $0 < \delta_0 < \min\{\frac{1}{2}, c_0/(1+c_0^2)\}$, $0 < \delta < \min\{\frac{1}{2}, c_2/(1+c_2^2)\}$ and large $K > 0$. Using (16)–(17), we get the following inequality:

$$\dot{E}(t) \leqslant -\omega E(t) \leqslant 0 \tag{25}$$

for some positive ω.

The above process could be also applied to compute

$$\mathrm{Re}\langle A(f,g,\phi,\psi),(f,g,\phi,\psi)\rangle \leqslant 0, \quad \forall (f,g,\phi,\psi) \in D(A).$$

So A is dissipative in H (Pazy, 1983), and (24) shows that if A generates a C_0-semigroup, then this semigroup must be exponentially stable. By the Lumer–Phillips theorem (Theorem 4.3, Pazy, 1983, p. 14), the proof will be accomplished if we can show that A^{-1} exists and is bounded on H. Actually, a simple computation shows that

$$A^{-1}(f,g,\phi,\psi) = (f^*, g^*, \phi^*, \psi^*), \quad \forall (f,g,\phi,\psi) \in H,$$

where $g^* = f$, $\psi^* = \phi$ and

$$\begin{aligned} f^*(x) = & c_0 f(0)(x-1) + \int_0^x (x-\tau)g(\tau)\,d\tau \\ & - \int_0^1 (1-\tau)g(\tau)\,d\tau,\end{aligned}$$

$$\begin{aligned}\phi^*(x) = & -\int_x^1 x\psi(\tau)\,d\tau - \int_0^x \tau\psi(\tau)\,d\tau \\ & - \frac{1}{c_1}\int_0^1 \psi(x)\,dx - c_2\phi(1)\left(x + \frac{1}{c_1}\right) \\ & + \frac{qf^*(0) + c_0 f(0)}{c_1}\left[1 - \frac{c_1+q}{q}(e^q - 1)(1 + c_1 x)\right].\end{aligned}$$

The proof is complete. □

5. Well-posedness and stability of closed-loop system

We go back to the closed-loop system (1) under the feedback (4):

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = -qw(0,t), \\ w(1,t) = \widehat{w}(1,t), \\ \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \\ \widehat{w}_x(0,t) = -qy(t) - c_0[\dot{y}(t) - \widehat{w}_t(0,t)], \\ \widehat{w}_x(1,t) = -c_2\widehat{w}_t(1,t) - (c_1+q)\widehat{w}(1,t) \\ \qquad - (c_1+q)\int_0^1 e^{q(1-\xi)}[c_2\widehat{w}_t(\xi,t)\,d\xi + q\widehat{w}(\xi,t)], \\ y(t) = w(0,t). \end{cases} \tag{26}$$

We consider system (26) in the state space $\mathscr{H} = \{(f,g,\phi,\psi) \in (H^1(0,1) \times L^2(0,1))^2 \mid f(1) = \phi(1)\}$. Define the system operator

$$\begin{cases} D(\mathscr{A}) = \{(f,g,\phi,\psi) \in \mathscr{H} \mid \mathscr{A}(f,g,\phi,\psi) \in \mathscr{H}, \\ f'(0) = -qf(0), \quad \phi'(0) = c_0\psi(0) - qf(0) - c_0 g(0), \\ \phi'(1) = -c_2\psi(1) - (c_1+q)\phi(1) \\ \qquad - (c_1+q)\int_0^1 e^{q(1-\xi)}[c_2\psi(\xi) + q\phi(\xi)]\,d\xi\}, \\ \mathscr{A}(f,g,\phi,\psi) = (g, f'', \psi, \phi''), \quad \forall (f,g,\phi,\psi) \in D(\mathscr{A}). \end{cases} \tag{27}$$

Then system (26) can be written as an evolution equation in $\mathscr{H}$:

$$\begin{aligned}&\frac{d}{dt}(w(\cdot,t), w_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)) \\ &\quad = \mathscr{A}(w(\cdot,t), w_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)).\end{aligned} \tag{28}$$

Theorem 2. *Let $\mathscr{A}$ be defined by* (27) *with $c_0, c_1, c_2 > 0$. Then $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ on $\mathscr{H}$. This semigroup is exponentially stable*:

$$\|e^{\mathscr{A}t}\| \leqslant Ce^{-\omega t}$$

for some positive constants C, ω independent of t.

Proof. For any $(w(\cdot,0), w_t(\cdot,0), \widehat{w}(\cdot,0), \widehat{w}_t(\cdot,0)) \in D(\mathscr{A})$, let

$$\begin{cases} \varepsilon(x,0) = w(x,0) - \widehat{w}(x,0), \\ \varepsilon_t(x,0) = w_t(x,0) - \widehat{w}_t(x,0), \\ \widetilde{w}(x,0) = \widehat{w}(x,0) + (c_1+q)\int_0^x e^{q(x-\xi)}\widehat{w}(\xi,0)\,d\xi, \\ \widetilde{w}_t(x,0) = \widehat{w}_t(x,0) + (c_1+q)\int_0^x e^{q(x-\xi)}\widehat{w}_t(\xi,0)\,d\xi. \end{cases}$$

A direct computation shows that the initial value $(\varepsilon(\cdot,0), \varepsilon_t(\cdot,0), \widetilde{w}(\cdot,0), \widetilde{w}_t(\cdot,0)) \in D(A)$, which implies that there exists a unique classical solution to (10). Let

$$\begin{cases} w(x,t) = \widetilde{w}(x,t) - (c_1+q)\int_0^x e^{-c_1(x-\xi)}\widetilde{w}(\xi,t)\,d\xi \\ \qquad + \varepsilon(x,t), \\ \widehat{w}(x,t) = \widetilde{w}(x,t) - (c_1+q)\int_0^x e^{-c_1(x-\xi)}\widetilde{w}(\xi,t)\,d\xi. \end{cases}$$

Then a direct computation shows that such a defined $(w, \widehat{w})$ satisfies (26) with initial value $(w(\cdot,0), w_t(\cdot,0), \widehat{w}(\cdot,0), \widehat{w}_t(\cdot,0))$. This solution is unique by the invertible transformation

$$\begin{pmatrix} \varepsilon \\ \varepsilon_t \\ \widetilde{w} \\ \widetilde{w}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & I+\mathbb{P} & 0 \\ 0 & 0 & 0 & I+\mathbb{P} \end{pmatrix} \begin{pmatrix} w \\ w_t \\ \widehat{w} \\ \widehat{w}_t \end{pmatrix} \tag{29}$$

and the uniqueness of classical solution to (10). Moreover, this solution is exponentially stable by (7), (13) and (29):

$$\begin{aligned} &\|(w(\cdot,t), w_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)\|_{\mathscr{H}} \\ &\quad \leqslant C\mathrm{e}^{-\omega t}\|(w(\cdot,0), w_t(\cdot,0), \widehat{w}(\cdot,0), \widehat{w}_t(\cdot,0)\|_{\mathscr{H}} \end{aligned} \tag{30}$$

for some positive constant C independent of t. Similar to the proof of Lemma 1, we can show that $\mathscr{A}^{-1}$ exists and is bounded on $\mathscr{H}$. Hence $\rho(\mathscr{A}) \neq \emptyset$. Since obviously, $D(\mathscr{A})$ is dense in $\mathscr{H}$, it follows from Theorem 1.3 of Pazy (1983, p. 102) that $\mathscr{A}$ generates a C_0-semigroup $\mathrm{e}^{\mathscr{A}t}$ on $\mathscr{H}$. (29) shows that $\mathrm{e}^{\mathscr{A}t}$ is exponentially stable. □

6. A collocated design

In this section, we design an observer which allows stabilization of the string with both measurement of displacement and actuation of vertical force at the right end, despite the presence of the destabilizing boundary condition at the left end as that in Eq. (1). This new controller requires the use of a more complicated, backstepping observer, as in Smyshlyaev and Krstic (2004). This new result is actually an extension of the classical theory, using the classical choices of actuation and sensing.

The reason that we have to use a more complicated observer in this new collocated result than in the previous non-collocated result is that for the non-collocated result the sensor was at the source of instability, the boundary condition of (1) at left end, whereas in this new result the sensor and the actuator are at the opposite end of the domain from the source of instability.

The model we are concerned with is the string equation of the following form:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = u(t), \\ y(t) = w(1,t), \end{cases} \tag{31}$$

where u is again a scalar control input and y is a scalar measured output. Note that if the string is pinned at $x=0$, namely $w(0,t)=0$, then one can use the classical derivative feedback $w_x(1,t)=-c_1 w_t(1,t)$. But now that the string is destabilized at $x=0$ due to the Robin boundary condition $w_x(0,t)=-qw(0,t)$, the more complicated backstepping controller, and a more complicated backstepping observer are employed.

We design the following observer:

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t) \\ \qquad + q(c_0+q)\mathrm{e}^{q(1-x)}[y(t)-\widehat{w}(1,t)] \\ \qquad + c_1(c_0+q)\mathrm{e}^{q(1-x)}[\dot{y}(t)-\widehat{w}_t(1,t)], \\ \widehat{w}_x(0,t) = -q\widehat{w}(0,t), \\ \widehat{w}_x(1,t) = u(t) + (c_0+q)[y(t)-\widehat{w}(1,t)] \\ \qquad + c_1[\dot{y}(t)-\widehat{w}_t(1,t)], \end{cases} \tag{32}$$

where c_i, $i=0,1,2,3$ are positive design parameters.

The observer error $\varepsilon(x,t)=w(x,t)-\widehat{w}(x,t)$ is governed by

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t) \\ \qquad - (c_0+q)\mathrm{e}^{q(1-x)}[c_1\varepsilon_t(1,t) + q\varepsilon(1,t)], \\ \varepsilon_x(0,t) = -q\varepsilon_t(0,t), \\ \varepsilon_x(1,t) = -c_1\varepsilon_t(1,t) - (c_0+q)\varepsilon(1,t). \end{cases} \tag{33}$$

The output feedback controller is designed as

$$\begin{aligned} u(t) = &-c_3\widehat{w}_t(1,t) - (c_2+q)\widehat{w}(1,t) \\ &- (c_2+q)\int_0^1 \mathrm{e}^{q(1-\xi)}[c_3\widehat{w}_t(\xi,t) + q\widehat{w}(\xi,t)]\,\mathrm{d}\xi. \end{aligned} \tag{34}$$

Then the closed-loop form of observer (32) becomes

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t) \\ \qquad + q(c_0+q)\mathrm{e}^{q(1-x)}[y(t)-\widehat{w}(1,t)] \\ \qquad + c_1(c_0+q)\mathrm{e}^{q(1-x)}[\dot{y}(t)-\widehat{w}_t(1,t)], \\ \widehat{w}_x(0,t) = -q\widehat{w}(0,t), \\ \widehat{w}_x(1,t) = -c_3\widehat{w}_t(1,t) - (c_2+q)\widehat{w}(1,t) \\ \qquad - (c_2+q)\int_0^1 \mathrm{e}^{q(1-\xi)}[c_3\widehat{w}_t(\xi,t) \\ \qquad + q\widehat{w}(\xi,t)]\,\mathrm{d}\xi \\ \qquad + (c_0+q)[y(t)-\widehat{w}(1,t)] \\ \qquad + c_1[\dot{y}(t)-\widehat{w}_t(1,t)]. \end{cases} \tag{35}$$

Make the invertible change of variables

$$\begin{aligned} \widetilde{\varepsilon}(x,t) &= [(I+\mathbb{P}_1)\varepsilon](x,t) \\ &= \varepsilon(x,t) - (c_0+q)\int_x^1 \mathrm{e}^{c_0(x-\xi)}\varepsilon(\xi,t)\,\mathrm{d}\xi, \\ \widetilde{w}(x,t) &= [(I+\mathbb{P}_2)\widehat{w}](x,t) \\ &= \widehat{w}(x,t) + (c_2+q)\int_0^x \mathrm{e}^{q(x-\xi)}\widehat{w}(\xi,t)\,\mathrm{d}\xi. \end{aligned}$$

Both $\mathbb{P}_1$ and $\mathbb{P}_2$ are Volterra transformations. Under these transformations, we obtain the following autonomous system:

$$\begin{cases} \widetilde{\varepsilon}_{tt}(x,t) = \widetilde{\varepsilon}_{xx}(x,t), \\ \widetilde{\varepsilon}_x(0,t) = c_0\widetilde{\varepsilon}(0,t), \quad \widetilde{\varepsilon}_x(1,t) = -c_1\widetilde{\varepsilon}_t(1,t), \\ \widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t) + \dfrac{c_0+q}{q}\mathrm{e}^{q}[c_1\widetilde{\varepsilon}_t(1,t) + q\widetilde{\varepsilon}(1,t)] \\ \qquad \times (q\cosh(qx) + c_2\sinh(qx)), \\ \widetilde{w}_x(0,t) = c_2\widetilde{w}(0,t), \\ \widetilde{w}_x(1,t) = -c_3\widetilde{w}_t(1,t) + c_1\widetilde{\varepsilon}_t(1,t) + (c_0+q)\widetilde{\varepsilon}(1,t). \end{cases} \tag{36}$$

Consider system (36) in the space $\mathscr{X} = (H^1(0,1)\times L^2(0,1))^2$ with the inner product

$$\begin{aligned} &\langle (f_1, g_1, \phi_1, \psi_1), (f_2, g_2, \phi_2, \psi_2)\rangle \\ &= c_0 f_1(0)\overline{f_2(0)} + K\int_0^1 [f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)}]\,\mathrm{d}x \\ &\quad + K\delta_0\int_0^1 (x+1)[f_1'(x)\overline{g_2(x)} + g_1(x)\overline{f_2'(x)}]\,\mathrm{d}x \\ &\quad + \int_0^1 [\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}]\,\mathrm{d}x + c_2\phi_1(0)\overline{\phi_2(0)} \\ &\quad + \delta\int_0^1 (x+1)[\phi_1'(x)\overline{\psi_2(x)} + \psi_1(x)\overline{\phi_2'(x)}]\,\mathrm{d}x, \end{aligned}$$

for all $(f, g, \phi, \psi) \in \mathscr{X}$, where δ_0, δ are small positive constants and $K > 0$ is large enough so that B is dissipative in $\mathscr{X}$ as in the proof of Lemma 3. Define the system operator $B : D(B)(\subset \mathscr{X}) \to \mathscr{X}$ as follows:

$$\begin{cases} D(B) = \{(f, g, \phi, \psi) \in (H^2(0,1) \times H^1(0,1))^2 | \\ f'(0) = c_0 f(0), \quad f'(1) = -c_1 g(1), \\ \phi'(0) = c_2 \phi(0), \\ \phi'(1) = -c_3 \psi(1) + c_1 g(1) + (c_0 + q) f(1)\}, \\ B(f, g, \phi, \psi) = \Big(g, f'', \psi, \phi'' + (c_0 + q) \\ \qquad \times [c_1 g(1) + q f(1)] e^q \Big(\cosh(qx) \\ \qquad + \dfrac{c_2}{q} \sinh(qx) \Big)\Big) \\ \qquad \forall (f, g, \phi, \psi) \in D(B). \end{cases} \tag{37}$$

Then system (36) can be written as

$$\frac{d}{dt} \mathbf{v}(\cdot, t) = B \mathbf{v}(\cdot, t), \tag{38}$$

where $\mathbf{v} = (\widetilde{\varepsilon}, \widetilde{\varepsilon}_t, \widetilde{w}, \widetilde{w}_t)$.

Lemma 3. *Let B be defined by* (37). *Then B generates an exponentially stable C_0-semigroup on $\mathscr{X}$. Therefore, for any initial value $\mathbf{v}(\cdot, 0) \in \mathscr{X}$, there exists a unique solution to* (38) *such that $\mathbf{v}(\cdot, t) \in C([0, \infty); \mathscr{X})$, and there are positive constants L, ω_0 such that*

$$\|\mathbf{v}(\cdot, t)\| \leqslant L e^{-\omega_0 t} \|\mathbf{v}(\cdot, 0)\|. \tag{39}$$

Moreover, if $\mathbf{v}(\cdot, 0) \in D(B)$, then $\mathbf{v}(\cdot, t) \in C^1([0, \infty); \mathscr{X})$ is the classical solution to (36).

Proof. The proof is very similar to the proof of Lemma 1, so we give just a sketch. Define the Lyapunov functions

$$E_{\widetilde{\varepsilon}}(t) = \frac{1}{2} \int_0^1 [\widetilde{\varepsilon}_x^2(x, t) + \widetilde{\varepsilon}_t^2(x, t)] \, dx + \delta_0 \int_0^1 (x+1) \widetilde{\varepsilon}_x(x, t) \widetilde{\varepsilon}_t(x, t) \, dx + \frac{c_0}{2} \widetilde{\varepsilon}^2(0, t), \tag{40}$$

$$E_{\widetilde{w}}(t) = \frac{1}{2} \int_0^1 [\widetilde{w}_x^2(x, t) + \widetilde{w}_t^2(x, t)] \, dx + \frac{c_2}{2} \widetilde{w}^2(0, t) + \delta \int_0^1 (1 + x) \widetilde{w}_x(x, t) \widetilde{w}_t(x, t) \, dx. \tag{41}$$

Both of them are positive definite for small positive δ_0, δ (which is shown as in Lemma 1). One can show that the derivative along the solution of (36) of the overall Lyapunov function $E(t) = K E_{\widetilde{\varepsilon}}(t) + E_{\widetilde{w}}(t)$ is

$$\begin{aligned} \dot{E}(t) \leqslant & -\frac{\delta}{4} \int_0^1 [\widetilde{w}_x^2(x, t) + \widetilde{w}_t^2(x, t)] \, dx - \frac{\delta}{4} c_2^2 \widetilde{w}^2(0, t) \\ & - \frac{\delta}{4} \widetilde{w}_t^2(0, t) - [c_3 - \delta(1 + c_3^2)] \widetilde{w}_t^2(1, t) \\ & - \left[K \frac{\delta_0}{2} - \Lambda \right] \int_0^1 \widetilde{\varepsilon}_x^2(x, t) \, dx - K \frac{\delta_0}{2} \int_0^1 \widetilde{\varepsilon}_t^2(x, t) \, dx \\ & - [K(c_0 - \delta_0(1 + c_0^2)) - \Lambda] \widetilde{\varepsilon}_t^2(1, t) \\ & - K \frac{\delta_0}{2} \widetilde{\varepsilon}_t^2(0, t) - \left[K \frac{\delta_0}{2} c_0^2 - \Lambda \right] \widetilde{\varepsilon}^2(0, t). \end{aligned} \tag{42}$$

Choosing δ_0, δ sufficiently small and K sufficiently large, there exists a sufficiently small $\omega_0 > 0$ such that

$$\dot{E}(t) \leqslant -\omega_0 E(t). \tag{43}$$

The rest of the proof is the same as in Lemma 1. □

Finally, we go back to the original closed-loop system (31), (32), (34) and consider it in space $\mathscr{X}$. Define the system operator

$$\begin{cases} D(\mathscr{B}) = \{(f, g, \phi, \psi) \in (H^2(0,1) \times L^2(0,1))^2 |, \\ f'(0) = -q f(0), \phi'(0) = -q \phi(0), \\ \phi'(1) = -c_3 \psi(1) - (c_2 + q) \phi(1) \\ \qquad - (c_2 + q) \int_0^1 e^{q(1-\xi)} [c_3 \psi(\xi) + q \phi(\xi)] \, d\xi, \\ f'(1) = \phi'(1) + (c_0 + q)[f(1) - \psi(1)] \\ \qquad + c_1 [g(1) - \psi(1)]\}, \\ \mathscr{B}(f, g, \phi, \psi) = (g, f'', \psi, \phi'' + (c_0 + q) e^{q(1-\cdot)} \\ \qquad \times [q(f(1) - \phi(1)) + c_1(g(1) - \psi(1))]) \\ \qquad \forall (f, g, \phi, \psi) \in D(\mathscr{B}). \end{cases} \tag{44}$$

Then the closed-loop system can be written as an evolution equation in $\mathscr{X}$:

$$\begin{aligned} & \frac{d}{dt} (w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t)) \\ & \quad = \mathscr{B}(w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t)). \end{aligned} \tag{45}$$

Theorem 4. *Let $\mathscr{B}$ be defined by* (44) *with $c_0, c_1, c_2, c_3 > 0$. Then $\mathscr{B}$ generates an exponentially stable C_0-semigroup $e^{\mathscr{B} t}$ on $\mathscr{X}$. This semigroup is exponentially stable*:

$$\|e^{\mathscr{B} t}\| \leqslant C_0 e^{-\omega_0 t}$$

for some positive constants C_0, ω_0 independent of t.

Proof. The result follows from Lemma 3 in the same way as Theorem 2 follows from Lemma 1 by noticing that the invertible transformation below relates the systems (36) and (44),

$$\begin{pmatrix} \widetilde{\varepsilon} \\ \widetilde{\varepsilon}_t \\ \widetilde{w} \\ \widetilde{w}_t \end{pmatrix} = \begin{pmatrix} I + \mathbb{P}_1 & 0 & -I - \mathbb{P}_1 & 0 \\ 0 & I + \mathbb{P}_1 & 0 & -I - \mathbb{P}_1 \\ 0 & 0 & I + \mathbb{P}_2 & 0 \\ 0 & 0 & 0 & I + \mathbb{P}_2 \end{pmatrix} \times \begin{pmatrix} w \\ w_t \\ \widehat{w} \\ \widehat{w}_t \end{pmatrix}. \quad \square$$

7. Frequency domain representation

With observers and controllers given explicitly, it is possible to obtain compensator transfer functions in closed form. The plant has the following transfer function:

$$y(s)=\frac{s}{s\cosh s-q\sinh s}u(s). \tag{46}$$

Let us first consider the non-collocated design (4), (5). Taking Laplace transform of (5) we get (the initial conditions of the observer are assumed to be zero):

$$s^2\hat{w}(x,s)=\hat{w}''(x,s),$$

$$\hat{w}'(0,s)=-(q+c_0s)y(s)+c_0s\hat{w}(0,s),$$

$$\begin{aligned}\hat{w}'(1,s)=&-(c_2s+c_1+q)\hat{w}(1,s)\\&-(c_1+q)(c_2s+q)\int_0^1 \mathrm{e}^{q(1-\xi)}\hat{w}(\xi,s)\,\mathrm{d}\xi.\end{aligned} \tag{47}$$

The solution to the first equation of (47) is

$$\hat{w}(x,s)=D_1\mathrm{e}^{sx}+D_2\mathrm{e}^{-sx}, \tag{48}$$

where D_1 and D_2 are constants that should be determined from the second and third equations of (47). The result is given by

$$D_1=-\frac{(q+c_0s)(s-q)(s(1-c_2)(s-c_1)\mathrm{e}^{-s}-\gamma)}{s[a(s)\mathrm{e}^s-b(s)\mathrm{e}^{-s}-\mathrm{e}^q(c_1+q)(c_2s+q)]}y(s),$$

$$D_2=-\frac{(q+c_0s)(s+q)(s(1+c_2)(s+c_1)\mathrm{e}^{s}-\gamma)}{s[a(s)\mathrm{e}^s-b(s)\mathrm{e}^{-s}-\mathrm{e}^q(c_1+q)(c_2s+q)]}y(s),$$

where we denote

$$a(s)=\tfrac{1}{2}(1+c_2)(1+c_0)s(s+q)(s+c_1),$$

$$b(s)=\tfrac{1}{2}(1-c_2)(1-c_0)s(s-q)(s-c_1).$$

The control input is $u(s)=D_1\mathrm{e}^s+D_2\mathrm{e}^{-s}=C(s)y(s)$, where

$$C(s)=\frac{P(s)}{Q(s)}, \tag{49}$$

$$\begin{aligned}P(s)=&(c_0s+q)[s^2(s+c_1c_2)+qs(c_2s+c_1)\\&-\mathrm{e}^q(c_1+q)(c_2s+q)(s\cosh(s)-q\sinh(s))],\end{aligned} \tag{50}$$

$$Q(s)=s(a(s)\mathrm{e}^s-b(s)\mathrm{e}^{-s}-\mathrm{e}^q(c_1+q)(c_2s+q)) \tag{51}$$

is the desired compensator transfer function. Note that zero is not a pole of (51) since $C(0)=q(1-q)-\mathrm{e}^{-q}qc_1/(c_1+q)$. The above explicit expression may be useful for finding an approximate, finite-dimensional reduced order model for the compensator. In Fig. 3 the Bode plots of the open-loop plant and the compensator are shown for $q=1$, $c_0=0.8$, $c_1=5$, $c_2=1$. It is clear that the compensator works by putting zeros close to the open-loop poles. Although this may not seem as a robust design, in Section 8 we will show that the compensator is robust with respect to parameter q in the plant.

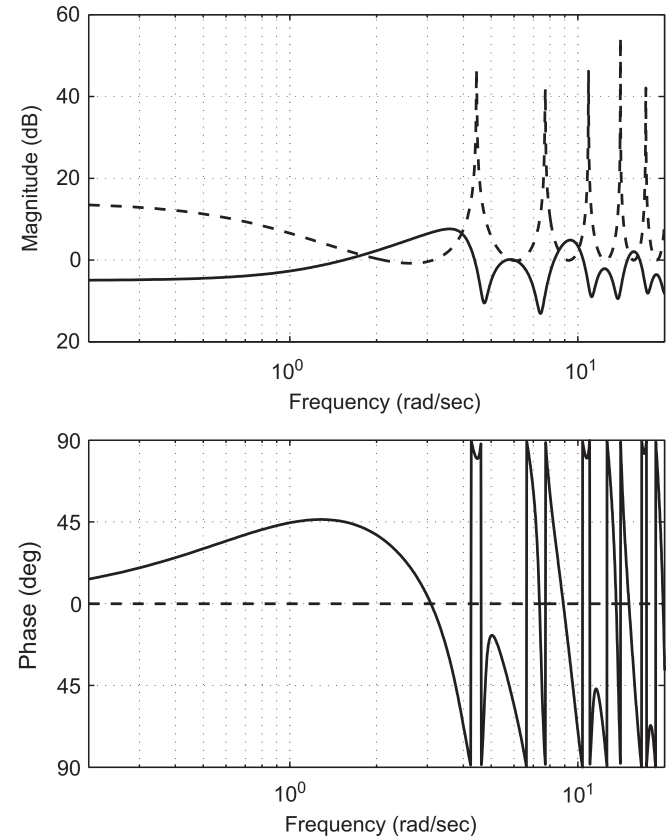

Fig. 3. Bode plots of the plant (dashed) and the compensator (solid) for the non-collocated design.

In a similar way one can derive the compensator transfer function in the collocated case:

$$\begin{aligned}P(s)=&\left(s(c_0+q)(c_1s+q)(c_3s+c_2)+sb(s)\frac{\mathrm{e}^{2q}-1}{2q}\right.\\&\left.-a(s)\right)[(s+q)\mathrm{e}^{-s}-(s-q)\mathrm{e}^s]-s^2(c_1s^2+c_0s+qs\\&+qc_0c_1)[(1+c_3)(s+c_2)\mathrm{e}^s-(1-c_3)(s-c_2)\mathrm{e}^{-s}],\end{aligned}$$

$$\begin{aligned}Q(s)=&s(s^2-q^2)[(c_0c_1+(1+c_3)(s+c_2))\mathrm{e}^s\\&-(c_0c_1+(1-c_3)(s-c_2))\mathrm{e}^{-s}]+2a(s)\cosh s\\&+2b(s)\sinh s-b(s)\frac{\mathrm{e}^{2q}-1}{2q}[(s-q)\mathrm{e}^s+(s+q)\mathrm{e}^{-s}],\end{aligned}$$

where $a(s)=s(s^2-q^2)(c_1s+c_0+q)$ and $b(s)=(c_0+q)(c_2+q)(c_1s+q)(c_3s+q)$.

8. Simulation results

In this section we consider a computational example for the non-collocated design (1), (2), (4). The second order in time equations were first converted into a system of two first order equations and then Backward Euler Method in time with Chebyshev spectral method in space were used. The boundary conditions were implemented in an explicit way. The numerical code was programmed in Matlab (see, e.g., Trefethen, 2000). We used spatial grid size $N=40$ and time step $\mathrm{d}t=10^{-4}$. The

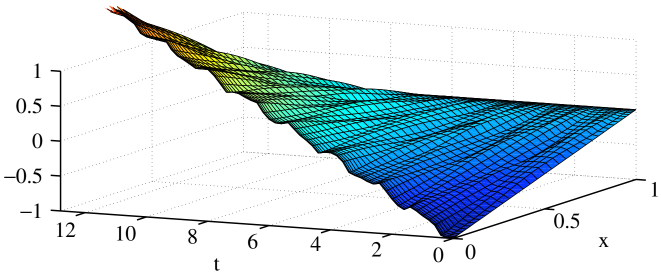

Fig. 4. String response $w(x,t)$. Uncontrolled case, zero Dirichlet boundary condition at $x=1$.

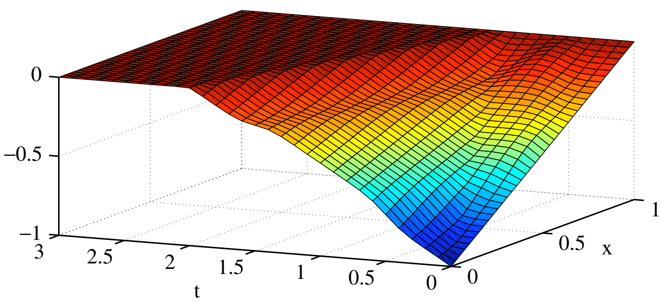

Fig. 5. Observer error $w-\widehat{w}$ in the uncontrolled case.

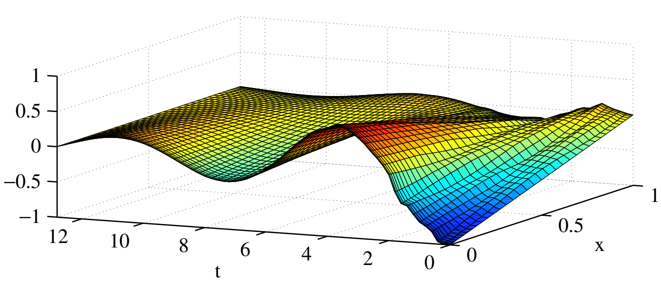

Fig. 6. String response $w(x,t)$ with observer-based control applied at $x=1$.

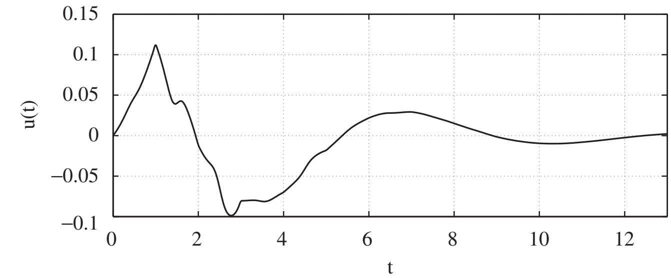

Fig. 7. Time trace of control $u(t)$.

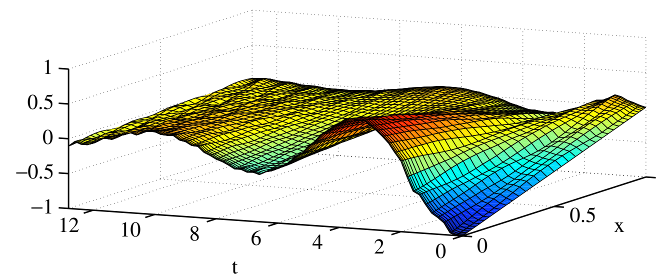

Fig. 8. String response $w(x,t)$ with reduced order ($N=6$) observer-based control applied at $x=1$.

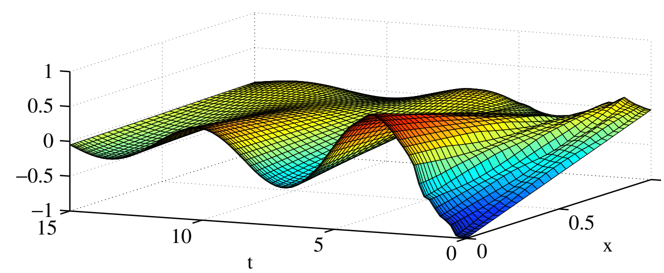

Fig. 9. String response $w(x,t)$ with $q=1$ in the observer mismatching the value $q=1.1$ in the plant.

parameter values were set to $q=1$, $c_0=1$, $c_1=600$, and $c_2=1$. The initial conditions were

$$w(0,x)=x-1 \quad \text{for } x\in[0,1],$$

$$w_t(0,x)=\begin{cases}1 & \text{if } 0.45\leqslant x\leqslant 0.55,\\ 0 & \text{otherwise}.\end{cases}$$

The initial condition on w corresponds to the simplest non-zero equilibrium profile of the uncontrolled string. The initial condition on the time derivative w_t corresponds to hitting the string out of its equilibrium with a force. The force is concentrated over a length of 0.1 in the middle of the string. As Fig. 4 shows, the forced boundary condition $w_x(0,t)=-qw(0,t)$ with zero input ($u\equiv 0$) and with zero Dirichlet boundary condition at $x=1$ results in the instability of the equilibrium profile. For the observer we used zero initial conditions, which correspond to the case when the observer initially assumes no knowledge of the plant. Fast decay of the observer error $w-\widehat{w}$ of the uncontrolled case can be seen in Fig. 5. The controlled case in Fig. 6 shows asymptotic stability. The instability at $x=0$ is suppressed with small control effort at $x=1$, shown in Fig. 7. We were able to achieve stabilization using reduced order observer with $N=6$ Chebyshev node points (keeping the plant discretization at $N=40$). The controlled plant is shown in Fig. 8 (compare it to Fig. 6).

In order to examine the robustness of our control/observer design with respect to the plant parameter q, we designed the compensator for $q=1$ and then changed the value of q in the plant. Our design was able to stabilize the system for values $q\in[0.5,1.2]$. The controlled system with $q=1$ used in the observer is depicted in Fig. 9. The closed loop poles are shown in Fig. 10. One can see that the controller is quite robust to the underestimation of the plant parameter by 10%.

9. Conclusions

We have presented the first control designs for stabilization of open-loop unstable wave equations. Previous damping-based boundary control designs were applicable only to wave equations that are neutrally stable (though not exponentially stable).

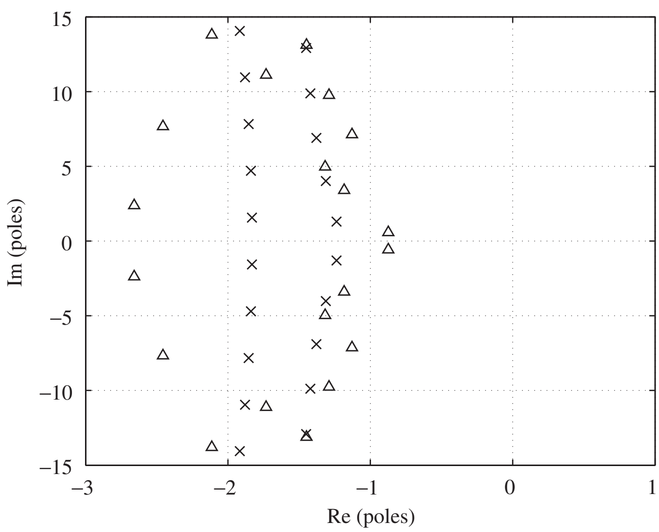

Fig. 10. Closed-loop poles: ×—nominal plant and compensator ($q = 1$), △—plant with $q = 1.1$, but compensator is designed for $q = 1$.

The continuing work is focused on extending the results of this work to beam models—shear beams and Timoshenko beams. Future work will deal with thin plates and cylindrical shells, with control applied along one edge and sensing applied along the other edge. Further efforts for hyperbolic PDEs will also include beam equations with non-constant coefficients (beams with spatially varying profiles).

Acknowledgments

This work was supported in part by National Science Foundation and in part by the Los Alamos National Laboratory.

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